On chaotic motion of a double pendulum

A. Aan, E. Aarend and M. Heinloo

Institute of Technology, Estonian University of Life Sciences, Kreutzwaldi 56, EE 51014 Tartu, Estonia, e-mail: aare.aan@emu.ee

Abstract. This paper presents the results of studying the chaotic motion of a virtual double pendulum. Large oscillations of this pendulum were modelled by the system of nonlinear differential equations in the Hamilton form. This system was solved on the worksheet of Computer Package Mathcad numerically, using the Runge-Kutta algorithm of fourth order. The results are illustrated by some frames of video clips visualizing the chaotic motion of a double physical pendulum. The results of this paper can be used in the teaching process of analytical mechanics for students of engineering.

Key words: Analytical mechanics, equations of Hamilton, double pendulum, chaotic motion, Computer Package Mathcad.

INTRODUCTION

Classical analytical mechanics contains a lot of differential equations in the Lagrange and the Hamilton forms. These equations, mostly nonlinear, are practically dead before their numerical solution and imagination of the results. It is convenient to use different computer programs for this purpose, for example, the Computer Package Mathcad. From the point of view of teaching it is useful to simulate, if possible, the motion of a virtual object, modelled by the nonlinear equations of Lagrange or Hamilton. Such an object is, for instance, a double pendulum. It appears that the motion of virtual and physical models of a double pendulum is chaotic. It is possible to find a lot of results on the study of chaotic virtual models of double pendulums on different web sites, for example [1]–[12]. The physical models of a double pendulum are presented, for example, on the web sites [13]–[18].

The purpose of this paper is to use the features of the Computer Package Mathcad and solve numerically, using the Runge-Kutta algorithm of fourth order, the nonlinear system of differential equations in the Hamilton form, modelling the large oscillation of a virtual physical double pendulum, and to visualize its chaotic motion for different initial conditions.

MATERIALS AND METHODS

A mathematical model to study the motion of a double pendulum

Let us consider a physical double pendulum in Fig. 1

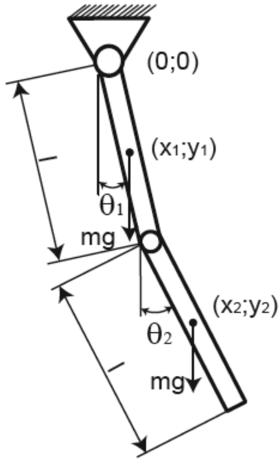

Figure 1. A physical double pendulum.

The double pendulum in Fig. 1 consists of two links of equal length and gravity mg, where g is the acceleration of gravity, joined together by a pivot. The upper pivot of the double pendulum is grounded. In the current position the upper link of the double pendulum has inclination angle θ_1 and the lower link has inclination angle θ_2. The co-ordinates of the centre of mass of the upper link are (x_1, y_1) and of the lower link – (x_2, y_2).

The mathematical model in the Hamilton form, describing large oscillations of the double pendulum is [18, 19]

$$\theta'_1 = \frac{6}{ml^2}\frac{2p_{\theta_1} - 3p_{\theta_2}\cos(\theta_1-\theta_2)}{16-9\cos(\theta_1-\theta_2)^2}, \qquad \theta'_2 = \frac{6}{ml^2}\frac{8p_{\theta_2} - 3p_{\theta_1}\cos(\theta_1-\theta_2)}{16-9\cos(\theta_1-\theta_2)^2}, \tag{1}$$

$$p'_{\theta_1} = -\frac{1}{2}ml^2\left(\theta'_1\theta'_2\sin(\theta_1-\theta_2) + 3\frac{g}{l}\sin(\theta_1)\right),$$

$$p'_{\theta_2} = -\frac{1}{2}ml^2\left(-\theta'_1\theta'_2\sin(\theta_1-\theta_2) + \frac{g}{l}\sin(\theta_2)\right),$$

where θ_1 and θ_2 are generalized co-ordinates [19], or inclination angles, of the double pendulum (Fig. 1) and $p_{\theta_1} = \frac{1}{6}ml^2(8\theta'_1 + 3\theta'_2\cos(\theta_1-\theta_2))$, $p_{\theta_2} = \frac{1}{6}ml^2(2\theta'_2 + 3\theta'_1\cos(\theta_1-\theta_2))$, are the generalized impulses [20] of the double pendulum. Note, that for small oscillations of a double pendulum, considered in [21], the mathematical model is following form equations (1), when $\cos(\theta_1-\theta_2)\approx 1$, $\sin(\theta_1-\theta_2)\approx 0$, $\sin(\theta_1)\approx 0$, $\sin(\theta_2)\approx 0$.

Let us assume that the special solution of the equations (1) satisfies the following initial conditions:

$$\theta_1(0)=\theta_{10}\,,\ \theta_2(0)=\theta_{20}\,,\ p_{\theta_1}(0)=p_{\theta_{10}}\,,\ p_{\theta_2}(0)=p_{\theta_{20}}\,, \tag{2}$$

where θ_{10}, θ_{10}, $p_{\theta_{10}}$, $p_{\theta_{20}}$ are constants. Let us consider the first initial position of the double pendulum, shown in Fig. 2 and assume that the motion of the double pendulum begins without initial angular velocities.

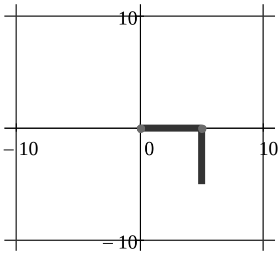

Figure 2. First initial position of the double pendulum.

This assumption means that in the initial conditions (2) the constants must have the following values:

$$\theta_{10}=\frac{\pi}{2}\,,\ \theta_{20}=0\,,\ p_{\theta_{10}}=0\,,\ p_{\theta_{20}}=0\,, \tag{3}$$

Let us consider now the initial position of the double pendulum, shown in Fig 3 and assume also that the motion of the double pendulum begins without initial angular velocities

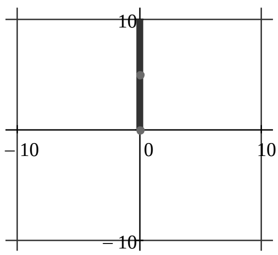

Figure 3. Second initial position of the double pendulum.

This assumption means that in the initial conditions there must be the following values of the constants:

$$\theta_{10}=\pi\,,\ \theta_{20}=\pi\,,\ p_{\theta_{10}}=0\,,\ p_{\theta_{20}}=0\,, \tag{4}$$

Numerical solution of the system of differential equations (1) on the worksheet of the Computer Package Mathcad

To use the algorithms programmed in Computer Package Mathcad for solution system (1) of differential equations on the worksheet of Computer Package Mathcad we have to use the following new variables with array subscripts $y_0 = \theta_1$, $y_1 = \theta_2$, $y_2 = p_{\theta_1}$, $y_3 = p_{\theta_2}$ and to present the system (1) in normal form

$$y'_0 = f_1(m, l, y)), \; y'_1 = f_2(m, l, y)), \; y'_2 = f_3(m, l, g, y)), \; y'_3 = f_4(m, l, g, y)), \qquad (5)$$

where: $f_1(m,l,y) = \dfrac{6}{ml^2}\dfrac{2y_2 - 3y_3\cos(y_0 - y_1)}{16 - 9\cos(y_0 - y_1)^2}$, $\quad f_2(m,l,y) = \dfrac{6}{ml^2}\dfrac{8y_3 - 3y_2\cos(y_0 - y_1)}{16 - 9\cos(y_0 - y_1)^2}$,

$$f_3(m,l,g,y) = -\frac{ml^2}{2}\left(\frac{36}{m^2l^4}\frac{[2y_2 - 3y_3\cos(y_0 - y_1)][8y_3 - 3y_2\cos(y_0 - y_1)]}{16 - 9\cos(y_0 - y_1)^4}\sin(y_0 - y_1) + \frac{3g}{l}\sin(y_0)\right),$$

$$f_4(m,l,g,y) = \frac{ml^2}{2}\left(\frac{36}{m^2l^4}\frac{[2y_2 - 3y_3\cos(y_0 - y_1)][8y_3 - 3y_2\cos(y_0 - y_1)]}{16 - 9\cos(y_0 - y_1)^4}\sin(y_0 - y_1) - \frac{g}{l}\sin(y_1)\right).$$

Here y is the vector $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$.

Computer Package Mathcad 14 [22] uses different functions, which presents certain numerical algorithms for solution systems of differential equations. One of them is the function rkfixed(a, t_0, t_N, N, D) , which uses the algorithm of Runge-Kutta. Here

$a = \begin{pmatrix} \theta_{10} \\ \theta_{20} \\ p\theta_{10} \\ p\theta_{20} \end{pmatrix}$ is the vector of constants in the initial conditions, t_0 is the first value of the time t and t_N – its last value; N is the number of intervals between t_0 and t_N; D – is the notation of the vector $D(t,y) = \begin{pmatrix} f_1(m,l,y \\ f_2(m,l,y \\ f_3(m,l,g,y \\ f_4(m,l,g,y \end{pmatrix}$ of the right hand sides of the system (5).

The function rkfixed(a, t_0, t_N, N, D) returns the matrix that is convenient to assign to another variable Z: Z = rkfixed(a, t_0, t_N, N, D). The first column with number 0 contains the values of the variable y_0 = t at the nodes. The second column with number 1 – the values of variable $y_1 = \theta_1$; the third column with number 2 – the values of variable $y_2 = \theta_2$; the fourth column with number 3 – the values of variable $y_3 = p_{\theta_1}$; the fifth column with number 4 – the values of variable $y_4 = p_{\theta_2}$.

RESULTS

The results of computations of $y_3 = p_{\theta_1}$ and $y_4 = p_{\theta_2}$ for $t_0 = 0$, $t_N = 100$, $N = 1{,}000$, $l = 5$, $m = 1$, $g = 9{,}807$ in the case of initial values (4) are given in the following tables, copied from the worksheet of the Computer Package Mathcad. These tables show that because of the approximate value of the $\pi = 3.141592653589793\ldots$, the motion of the double pendulum is possible even from the initial values (4).

$p_{\theta_1} =$

	0
0	0
1	$-9.00728588395204 \cdot 10^{-16}$
2	$-1.8014571767904077 \cdot 10^{-15}$
3	$-2.7021857651856114 \cdot 10^{-15}$
4	$-3.6029143535808155 \cdot 10^{-15}$
5	$-4.503642941976019 \cdot 10^{-15}$
6	$-5.404371530371223 \cdot 10^{-15}$
7	$-6.305100118766426 \cdot 10^{-15}$
8	$-7.205828707161631 \cdot 10^{-15}$
9	$-8.106557295556835 \cdot 10^{-15}$
10	$-9.007285883952038 \cdot 10^{-15}$
11	$-9.908014472347242 \cdot 10^{-15}$
12	$-1.0808743060742447 \cdot 10^{-14}$
13	$-1.170947164913765 \cdot 10^{-14}$
14	$-1.2610200237532854 \cdot 10^{-14}$
15	...

$p_{\theta_2} =$

	0
0	0
1	$-9.00728588395204 \cdot 10^{-16}$
2	$-1.8014571767904077 \cdot 10^{-15}$
3	$-2.7021857651856114 \cdot 10^{-15}$
4	$-3.6029143535808155 \cdot 10^{-15}$
5	$-4.503642941976019 \cdot 10^{-15}$
6	$-5.404371530371223 \cdot 10^{-15}$
7	$-6.305100118766426 \cdot 10^{-15}$
8	$-7.205828707161631 \cdot 10^{-15}$
9	$-8.106557295556835 \cdot 10^{-15}$
10	$-9.007285883952038 \cdot 10^{-15}$
11	$-9.908014472347242 \cdot 10^{-15}$
12	$-1.0808743060742447 \cdot 10^{-14}$
13	$-1.170947164913765 \cdot 10^{-14}$
14	$-1.2610200237532854 \cdot 10^{-14}$
15	...

Figs. 4 and 5 show the dependence of the inclination angles θ_1 and θ_2 of links of the double pendulum (Fig. 1) on time in the case of the initial conditions (3) (Fig. 2).

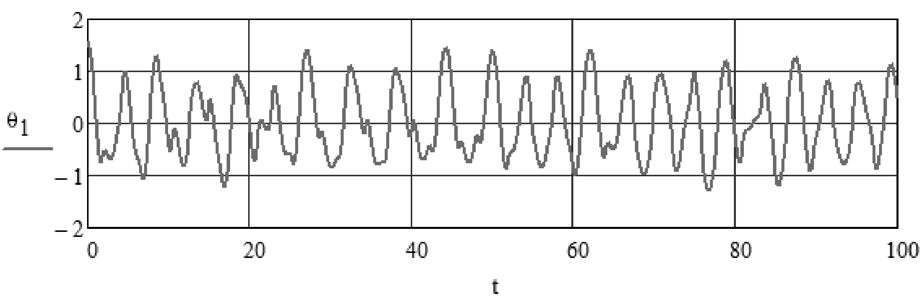

Figure 4. The dependence of inclination angle θ_1 on time in the case of initial values (3).

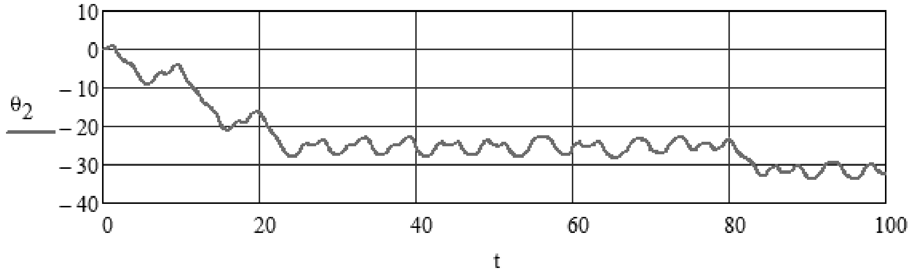

Figure 5. The dependence of inclination angle θ_2 on time in the case of initial values (3).

Figs. 6 and 7 show the dependence of the inclination angles θ_1 and θ_2 of links of the double pendulum (Fig. 1) on time in the case of the initial conditions (4) (Fig. 3).

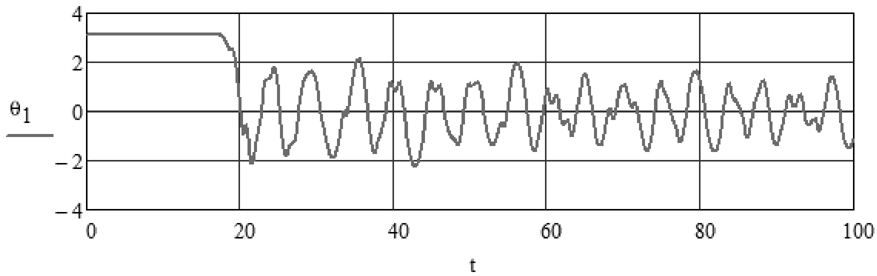

Figure 6. The dependence of inclination angle θ_1 on time in the case of initial values (4).

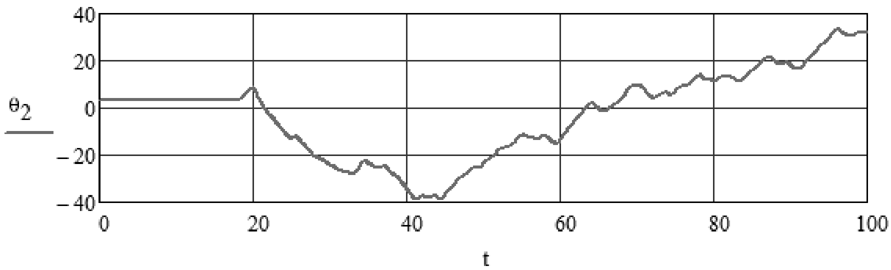

Figure 7. The dependence of inclination angle $\theta_2$on time in the case of initial values (4).

The diagrams in Figs. 4-7 show that there is chaotic motion of the double pendulum. The video clip http://www.youtube.com/watch?v=BFIJTAzYdvo shows a series of frames from this video clip.

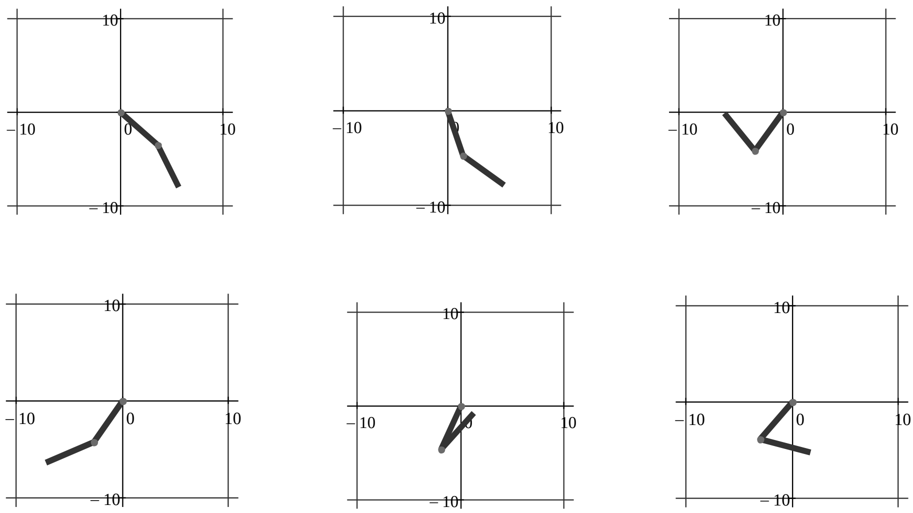

Figure 8. Frames from composed video clip from initial position in Fig. 2.

The video clip http://www.youtube.com/watch?v=N9eWZR3I-VQ shows the chaotic motion of the double pendulum from initial position in Fig. 3. Fig. 9 shows a series of frames from this video clip

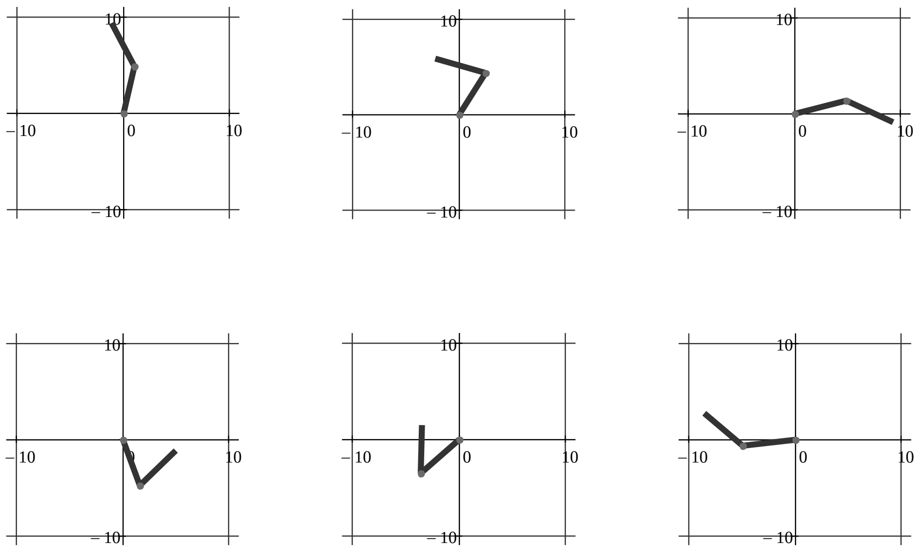

Figure 9. Frames from composed video clip from initial position in Fig. 3.

CONCLUSIONS

This paper shows the features of the Computer Package Mathcad in solution of nonlinear system of differential equations. The nonlinear equations of large oscillations of a double pendulum are solved and the chaotic motion is found out. Note that it is easy to make a physical model of a chaotic double pendulum. Some examples are given in [13]–[18]. Fig. 10 shows a frame from video clip with the chaotic motion of a simple physical pendulum [13].

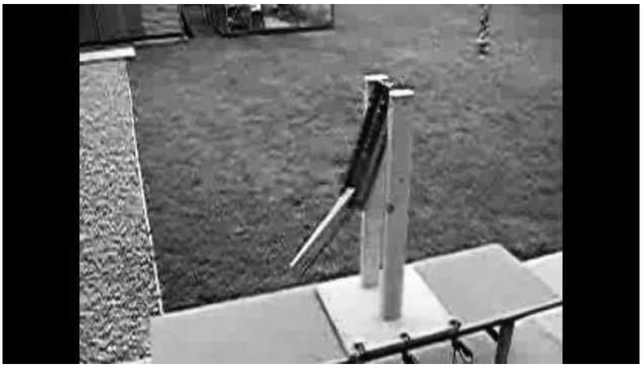

Figure 10. A frame from video clip, showing chaotic oscillation of the physical model of a double pendulum.

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