# Composing and solving differential equations for small oscillations of mathematical spring-coupled pendulums

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**Abstract.** This paper shows how to compose differential equations describing the oscillations of mathematical spring-coupled pendulums in a Mathcad worksheet. The composed differential equations are solved using both symbolic calculation features of Mathcad and Laplace transform. On the basis of the obtained solution, a video clip is made to demonstrate the motion of mathematical spring-coupled pendulums.
The methods presented in this paper can be used for teaching engineering mathematics and mechanics.

**Key words:** Mechanics, mechanical systems, small oscillations, a coupled pendulum, symbolic calculations, Laplace transform, simulation of motion.

## INTRODUCTION

Small oscillations of coupled pendulums are a classical problem in mechanical systems (Targ, 1976; Pain, 2005). The scheme of two mathematical pendulums coupled by a spring is shown in Fig 1.
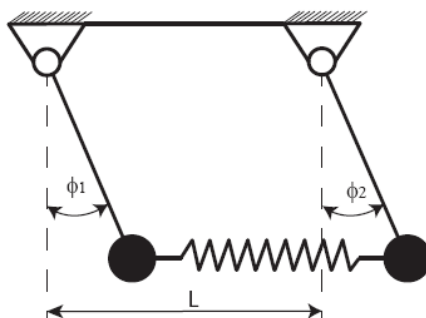


**Figure 1.** Mathematical spring-coupled pendulums.

Two mathematical spring-coupled pendulum can oscillate in the same direction $\phi_1 = \phi_2$ or in the opposite direction $\phi_1 = -\phi_2$. In these cases there is no energy transformation between the pendulums. In the beating case energy transfer occurs between the pendulums (Pain, 2005).

A paper (Picciarelli & Stella, 2010) shows how to use coupled pendulums as a physical system for laboratory investigations in upper secondary school. The students are actively engaged in simple model building, its implementation with an Excel

spreadsheet and comparing the measurements of the system behaviour with predictions from the model.

Numerous results of the study of coupled pendulums are presented on different web sites, such as Huisman & Fasolino, 2001; Hart, 2004; Christian, 2006; Fendt, 2010; Russel, 2012.

The present paper mainly focuses on the composition and solution of differential equations in a Mathcad worksheet, describing the motion of spring-coupled mathematical pendulums. The composed equations of motion have been solved in a Mathcad worksheet by using symbolic calculation features and Laplace transform.

First of all, the method and results presented in this paper can be implemented in the teaching process of engineering subjects, such as how to solve, compare and visualize mechanical problems using the computer.

## MATERIALS AND METHODS

**Composing a nonlinear system of differential equations for modeling large oscillations of mathematical coupled pendulum**

Let us consider two mathematical pendulums (Fig. 1) with equal length $l$ and masses $m$ that are coupled by a spring, with rigidity $c$. The distance between the pendulum's pivots is $L$ and inclination angles are $\phi_1$ and $\phi_2$.

The coordinates of the material points can be found out by the equations

$$
\begin{aligned}
x_1(t) &= l \cdot \sin(\phi_1(t)), \\
y_1(t) &= -l \cdot \cos(\phi_1(t)), \\
x_2(t) &= l \cdot \sin\big(\phi_2(t)\big) + L, \\
y_2(t) &= -l \cdot \cos(\phi_2(t)).
\end{aligned}
\tag{1}
$$

In order to compose a system of differential equations of motion for the coupled pendulums in Fig. 1, it is convenient to use the Mathcad features of symbolic calculations. Fig. 2 shows an automatic simplification of the formula for determining the kinetic energy of coupled pendulums in a Mathcad worksheet by using formulas (1) and the built-in function 'substitute' to automatically replace the first derivations of angles $\phi_1$ and $\phi_2$ at the time $t$ with $\phi'_1$ and $\phi'_2$.

$$
T = \frac{m \cdot \left[\left(\frac{d}{dt}x_1\right)^2 + \left(\frac{d}{dt}y_1\right)^2\right] + m \cdot \left[\left(\frac{d}{dt}x_2\right)^2 + \left(\frac{d}{dt}y_2\right)^2\right]}{2} \begin{vmatrix} \text{substitute}, \frac{d}{dt}\phi_1(t) = \phi'_1 \\ \text{substitute}, \frac{d}{dt}\phi_2(t) = \phi'_2 \end{vmatrix} = \frac{l^2 \cdot m \cdot \left(\phi'_1{}^2 + \phi'_2{}^2\right)}{2}
$$

**Figure 2.** Automatic derivation of the formula for determining kinetic energy in the Mathcad worksheet.

According to Fig. 2, the kinetic energy of the coupled pendulum is

$$
T = l^2 \cdot m \frac{\left(\phi'_1{}^2 + \phi'_2{}^2\right)}{2},
\tag{2}
$$

where $\phi'_1$ and $\phi'_2$ denote the first derivatives of $\phi_1$ and $\phi_2$.

The potential energy of the coupled pendulum is

$$\Pi = c \cdot l^2 \left(1 - \frac{\cos \phi_1{}^2}{2} - \frac{\cos \phi_2{}^2}{2} - \sin \phi_1 \cdot \sin \phi_2\right) - g \cdot l \cdot m(\cos \phi_1 - \cos \phi_2). \quad (3)$$

Substitution of (2) and (3) into Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\phi_1} + \frac{\partial \Pi}{\partial \phi_1} = 0, \frac{d}{dt}\frac{\partial T}{\partial \dot\phi_2} + \frac{\partial \Pi}{\partial \phi_2} = 0 \quad (4)$$

and simplification gives the following nonlinear system of coupled pendulum equations of motion:

$$\phi''_1 + \frac{g \cdot \sin \phi_1}{l} + \frac{c \cdot \cos \phi_1 \cdot \sin \phi_1}{m} - \frac{c \cdot \cos \phi_1 \cdot \sin \phi_2}{m} = 0, \quad (7)$$

$$\phi''_2 + \frac{g \cdot \sin \phi_2}{l} + \frac{c \cdot \cos \phi_2 \cdot \sin \phi_1}{m} - \frac{c \cdot \cos \phi_2 \cdot \sin \phi_2}{m} = 0,$$

where $\phi''_1$ and $\phi''_2$ are second derivatives of $\phi_1$ and $\phi_2$ describing large oscillations of the coupled pendulums in Fig 1.

**Symbolic solutions of linearized equations**
Let us consider small oscillations of coupled pendulums, when $\sin \phi \cong \phi$ and $\cos \phi \cong 1$. These equalities linearize nonlinear equations (7) and can be written in the forms

$$a \cdot \phi''_1 + b \cdot \phi_1 - d \cdot \phi_2 = 0, \quad (8)$$

$$a \cdot \phi''_2 - d \cdot \phi_1 + b \cdot \phi_2 = 0,$$

where $a = l^2 \cdot m$, $b = g \cdot l \cdot m + c \cdot l^2$ and $d = c \cdot l^2$.

According to Lepik & Roots, 1971, symbolic solution of equation systems (8) can be searched in the form

$$\phi_1(t, \lambda, K_1) = K_1 \cdot e^{\lambda \cdot t} \quad (9)$$

$$\phi_2(t, \lambda, K_2) = K_2 \cdot e^{\lambda \cdot t},$$

where $K_1$ and $K_2$ are constants. Substituting (9) into equations (8) and dividing by $e^{\lambda \cdot t}$ leads to the ordinary system of linear equations

$$(a \cdot \lambda^2 + b) \cdot K_1 - K_2 \cdot d = 0, \quad (10)$$

$$-K_1 \cdot d + (a \cdot \lambda^2 + b) \cdot K_2 = 0,$$

relative to $K_1$ and $K_2$. If the solution of the system (10) is not trivial ($K_1 = K_2 = 0$), then its determinant must be equal to zero:

$$\begin{vmatrix} (a \cdot \lambda^2 + b) & -d \\ -d & (a \cdot \lambda^2 + b) \end{vmatrix} = (a \cdot \lambda^2 + b)^2 - d^2 = 0. \quad (11)$$

Under condition (11), the solution of the equation system (10) is

$$K_1(\lambda) = \frac{a \cdot \lambda^2 + b}{d}, \quad K_2 = 1. \quad (12)$$

Equation (11) has four solutions:

$$\lambda_1 = i\,\omega_1, \quad \lambda_2 = -i\,\omega_1, \quad \lambda_3 = i\,\omega_2, \quad \lambda_4 = -i\,\omega_2, \tag{13}$$

where $\omega_1 = \sqrt{\dfrac{b-d}{a}}$, $\omega_2 = \sqrt{\dfrac{b+d}{a}}$ and $i = \sqrt{-1}$ – the imaginary unit. Substitution (13) into (12) gives $K_1(\lambda_1) = 1$, $K_1(\lambda_2) = 1$, $K_1(\lambda_3) = -1$, $K_1(\lambda_4) = -1$. Thus, the general solution of the system of differential equations (8) has the form

$$\phi_1(t, C_1, C_2, C_3, C_4) = C_1 \cdot e^{i\,\omega_1 \cdot t} + C_2 \cdot e^{-i\,\omega_1 \cdot t} - C_3 \cdot e^{i\,\omega_2 \cdot t} - C_4 \cdot e^{-i\,\omega_2 \cdot t}, \tag{14}$$

$$\phi_2(t, C_1, C_2, C_3, C_4) = C_1 \cdot e^{i\,\omega_1 \cdot t} + C_2 \cdot e^{-i\,\omega_1 \cdot t} + C_3 \cdot e^{i\,\omega_2 \cdot t} + C_4 \cdot e^{-i\,\omega_2 \cdot t},$$

where $i$–imaginary unit and $C_1$, $C_2$, $C_3$, $C_4$ are constants. By using Euler's famous formulas (Mathews & Howell, 2000),

$$e^{i\,\omega \cdot t} = \cos(\omega \cdot t) + i\sin(\omega \cdot t), \quad e^{-i\,\omega \cdot t} = \cos(\omega \cdot t) - i\sin(\omega \cdot t),$$

the real and imaginary parts of left sides of formulas (14) can be separated. Let us denote below the real parts of the general solution in form (14) also as

$$\phi_1(t, C_1, C_2, C_3, C_4) = (C_1 + C_2)\cos(\omega_1 \cdot t) - (C_3 + C_4)\cos(\omega_2 \cdot t), \tag{15}$$

$$\phi_2(t, C_1, C_2, C_3, C_4) = (C_1 + C_2)\cos(\omega_1 \cdot t) + (C_3 + C_4)\cos(\omega_2 \cdot t),$$

because these parts are also the general solution to the system of differential equations (8). Let us take the initial conditions in the form

$$\phi_1(0, C_1, C_2, C_3, C_4) = 0, \quad \phi_2(0, C_1, C_2, C_3, C_4) = \phi_{02}, \tag{16}$$
$$\phi'_1(0, C_1, C_2, C_3, C_4) = 0, \quad \phi'_2(0, C_1, C_2, C_3, C_4) = 0,$$

where

$$\phi'_1(t, C_1, C_2, C_3, C_4) = \frac{d}{dt}\phi_1(t, C_1, C_2, C_3, C_4),$$
$$\phi'_2(t, C_1, C_2, C_3, C_4) = \frac{d}{dt}\phi_2(t, C_1, C_2, C_3, C_4).$$

The solution to the system (16) relative to $C_1, C_2, C_3, C_4$ was found out by using the solve block (Fig. 3) in the Mathcad worksheet.

Given

$$\phi_1(0, C_1, C_2, C_3, C_4) = 0 \qquad \phi_2(0, C_1, C_2, C_3, C_4) = \phi_{02}$$
$$\phi'_1(0, C_1, C_2, C_3, C_4) = 0 \qquad \phi'_2(0, C_1, C_2, C_3, C_4) = 0$$

$$\mathrm{Find}(C_1, C_2, C_3, C_4) \rightarrow \begin{pmatrix} \dfrac{\phi_{02}}{2} \\ 0 \\ \dfrac{\phi_{02}}{2} \\ 0 \end{pmatrix}$$

**Figuere 3.** Solve block in the Mathcad worksheet.

Solve block in Fig. 3 gives $C_1(\phi_{02}) = \frac{\phi_{02}}{2}$, $C_2 = 0$, $C_3(\phi_{02}) = \frac{\phi_{02}}{2}$, $C_4 = 0$.

Taking into account these values, the general solution (15) takes the final form

$$\phi_1(t, \phi_{02}) = \frac{\phi_{02}}{2}(cos(\omega_1 \cdot t) - cos(\omega_2 \cdot t)),$$

$$\phi_2(t, \phi_{02}) = \frac{\phi_{02}}{2}(cos(\omega_1 \cdot t) + cos(\omega_2 \cdot t)). \tag{17}$$

**Solutions of linearized equations by using Laplace transform**

Another possibility to solve the system of differential equations (8) is to use the Laplace transform (or L-transform) (Mathews & Howell, 2000). Let us apply the Laplace transform to the differential equations (8)

$$a \cdot L(\phi''_1(t)) + b \cdot L(\phi_1(t)) - d \cdot L(\phi_2(t)) = 0, \tag{18}$$

$$a \cdot L(\phi''_2(t)) - d \cdot L(\phi_1(t)) + b \cdot L(\phi_2(t)) = 0.$$

By using the Laplace images (Mathews & Howell, 2000),

$$L(\phi''_1(t)) = s^2 \cdot \Phi_1(s) - s \cdot \phi_1(0) - \phi'_1(0), \; L(\phi_1(t)) = \Phi_1(s),$$
$$L(\phi''_2(t)) = s^2 \cdot \Phi_2(s) - s \cdot \phi_2(0) - \phi'_2(0), \; L(\phi_2(t)) = \Phi_2(s)$$

and taking into account the initial conditions

$$\phi_1(0) = 0, \; \phi_2(0) = \phi_{02}, \; \phi'_1(0) = 0, \; \phi'_2(0) = 0$$

the equations (18) lead to linear equations relative to the Laplace images $\Phi_1(s)$ and $\Phi_2(s)$

$$a \cdot s^2 \cdot \Phi_1(s) + b \cdot \Phi_1(s) - d \cdot \Phi_2(s) = 0, \tag{19}$$

$$a \cdot (s^2 \cdot \Phi_2(s) - s \cdot \phi_{02}) - d \cdot \Phi_1(s) + b \cdot \Phi_2(s) = 0.$$

Fig. 4 shows the solution to the system (19) by the solve block in the Mathcad worksheet

Given

$$a \cdot s^2 \cdot \Phi_1 + b \cdot \Phi_1 - d \cdot \Phi_2 = 0$$

$$a \cdot \left(s^2 \cdot \Phi_2 - s \cdot \phi_{02}\right) - d \cdot \Phi_1 + b \cdot \Phi_2 = 0$$

$$Find(\Phi_1, \Phi_2) \rightarrow \begin{pmatrix} \dfrac{a \cdot d \cdot s \cdot \phi_{02}}{a^2 \cdot s^4 + 2 \cdot a \cdot b \cdot s^2 + b^2 - d^2} \\ \dfrac{\phi_{02} \cdot a^2 \cdot s^3 + b \cdot \phi_{02} \cdot a \cdot s}{a^2 \cdot s^4 + 2 \cdot a \cdot b \cdot s^2 + b^2 - d^2} \end{pmatrix}$$

**Figure 4.** Solution to the system for determination of images in Mathcad worksheet.

Solve block in Fig. 4 gives

$$\Phi_1(s, \phi_{02}) = \frac{a \cdot d \cdot s \, \phi_{02}}{a^2 \cdot s^4 + 2 \, a \cdot b \cdot s^2 + b^2 - d^2},$$

$$\Phi_2(s, \phi_{02}) = \frac{(a^2 \cdot s^3 + a \cdot b \cdot s) \, \phi_{02}}{a^2 \cdot s^4 + 2 \, a \cdot b \cdot s^2 + b^2 - d^2}.$$

The solution to the system of differential equations (8) can be found out by inverse Laplace transform in the Mathcad worksheet (Fig. 5).

$$\phi_1(t, \phi_{02}) := \Phi_1(s, \phi_{02}) \; \text{invlaplace} \; \rightarrow \frac{\phi_{02} \cdot \cos\left(t \cdot \sqrt{\dfrac{b - d}{a}}\right)}{2} - \frac{\phi_{02} \cdot \cos\left(t \cdot \sqrt{\dfrac{b + d}{a}}\right)}{2}$$

$$\phi_2(t, \phi_{02}) := \Phi_2(s, \phi_{02}) \; \text{invlaplace} \; \rightarrow \frac{\phi_{02} \cdot \left(\cos\left(t \cdot \sqrt{\dfrac{b + d}{a}}\right) + \cos\left(t \cdot \sqrt{\dfrac{b - d}{a}}\right)\right)}{2}$$

**Figure 5.** Inverse Laplace transform in the Mathcad worksheet.

Fig 5 shows that the solution obtained here is identical with solution (17).

## RESULTS AND DISCUSSION

Let us assume that the coupled pendulums in Fig. 1 have the following parameters:

$$l = 2m, \; m = 5kg, \; c = 1N(m)^{-1}, \; L = 4m, \; g = 9.807m(s)^{-2}. \qquad (20)$$

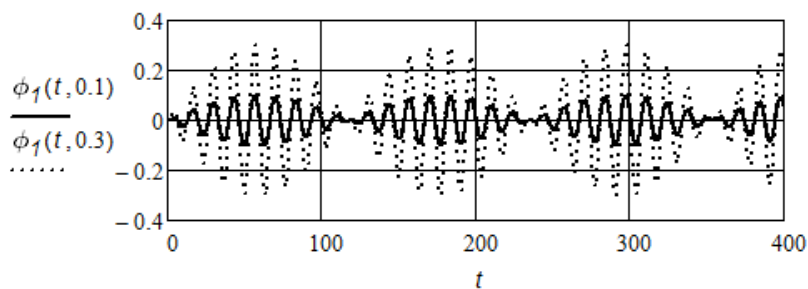Fig. 6 and 7 show the beating case of oscillation of double pendulums.



**Figure 6.** Dependence of angle $\phi_1(t, \phi_{02})$ on the time $t$ in the beating case for different values of $\phi_{02}$ in radians.
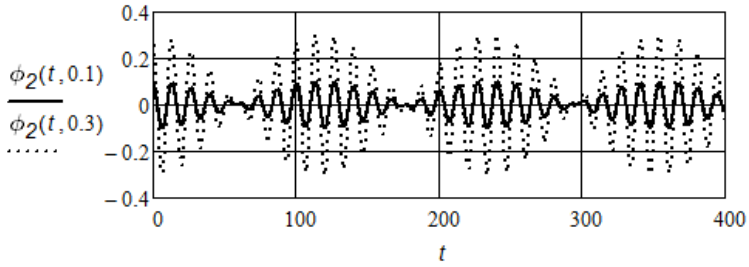
**Figure 7.** Dependence of angle $\phi_2(t, \phi_{02})$ on the time $t$ in the beating case for different values of $\phi_{02}$ in radians.

By using obtained here solution it is easy to compose, on the worksheet of Mathcad, the virtual model of a coupled pendulum and to simulate its motion. Fig. 8 shows some frames of composed video clip (Aan, 2011), showing clearly the beating case of motion of coupled pendulums. First frame in Fig. 8 shows the initial positions of coupled pendulums, used in computations.
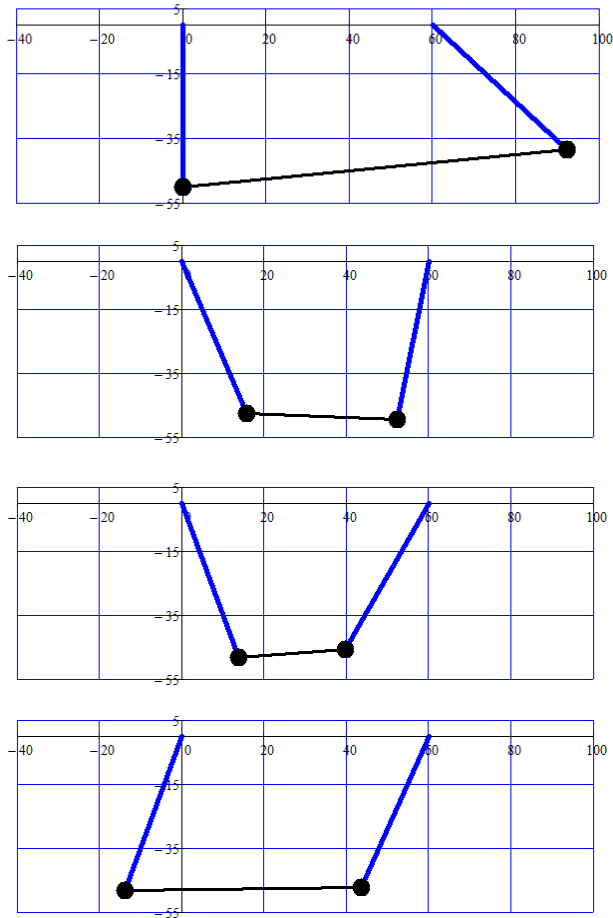


**Figure 8.** Some frames form, composed video clip (Aan, 2011).

## CONCLUSIONS

This paper demonstrates that the Mathcad is a suitable tool for solving problems of small oscillations using both the features of symbolic calculations and the Laplace transform.

It has been shown that the solutions of linearized system of differential equation of motion, obtained by symbolic calculation and by Laplace transform are coincides that is important from point of teaching of engineering mathematics.

Composed video clip shows clearly the motion of the pendulums, coupled by s spring, in the beating case that is important from point of teaching oscillations in engineering mechanics.

It is interesting to study also in future the oscillations of the pendulums, coupled by a torsion spring.

Methods presented in this paper can be used in teaching engineering mathematics and mechanics.

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