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Theory of vibration-assisted sugar beet root lifting

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Abstract. The vibration-assisted lifting of sugar beet roots from the soil has been gaining increasingly wide use worldwide and the majority of sugar beet harvesting machinery manufacturers produce beet harvesters equipped with just such kind of lifting units. In such units the priorities are low tractive resistance, the high quality of harvesting in terms of undamaged side surfaces of beet root bodies and intact tail parts as well as the high degree of their initial cleaning from the stuck soil. However, the parameters of the oscillatory processes generated by the vibrational lifting units used on the majority of sugar beet harvesting machinery in the market have rather average values appropriate for relatively favourable harvesting conditions (soft loose soil, beet root sizes close to the average, properly lined up planting rows etc.). But when the harvesting conditions deviate from their favourable values (especially in case of dry and strong soil), the vibrating lifters start performing the digging process with significant damage to the beet roots (breaking and tearing off the tail parts), their power consumption rises excessively sharply, the unit vibration drives prove to be unreliable. The literature source analysis has shown that any sufficiently detailed, comprehensive and dependable theory of direct beet root lifting from the soil is virtually absent. Thus, the aim of this research study has been to work out such a theoretical basis for the process of vibration-assisted beet root lifting, which will allow to calculate, in accordance with the harvesting parameters, the optimal design and kinematic parameters of the process ensuring the high quality of harvesting. A new theory has been developed, which describes the process of direct vibration-assisted beet root lifting performed under the effect of the vertical disturbing force and the pulling force, imparted to the root by the lifting unit. The obtained system of differential equations has made it possible to establish the law of motion of the beet root in the process of its direct vibration-assisted lifting and perform PC-based numerical calculations, which provide the basis for determining optimal kinematic modes of operation and design parameters of vibrational lifting units subject to the condition of maintaining sugar beet roots intact when harvesting them.

Key words: harvesting machinery, sugar beet root, vibration, lifting unit, modelling, elastic medium, differential equations.

INTRODUCTION

The harvesting of sugar beet roots with the use of vibrational lifting units has a number of advantages in comparison with other methods of digging them out (Sarec et. al., 2009; Lammers, 2011; Lammers & Schmittmann, 2013; Gu et al., 2014). For example, this way of retrieving sugar beet roots from the soil exhibits a lesser extent of root body damage and loss, a relatively low draught resistance and the more intensive clean-up of the root side surfaces from the adhering soil already at the lifting stage.

However, the said advantages of the vibrational method of beet root lifting are observed only under relatively favourable harvesting conditions, when the soil in the beet plantation is medium strong and not dry (especially at the depth of unit running in the soil). In addition, the beet root planting rows need be straight and the root body sizes – close to the average values (Vermeulen & Koolen, 2002).

Hence, the further research into the work process of the vibration-assisted lifting of beet roots from the soil and the development of improved lifting units basing on the results of such research is one of the topical challenges in the sugar beet growing industry (Gruber, 2005).

Problem. Fundamental theoretical research into the vibration-assisted beet root lifting enables the scientific substantiation of the design and kinematic parameters of vibrational lifting units. Such research is needed first of all for the theoretical analysis of the operation of vibrational lifting units specifically in unfavourable harvesting conditions, on heavy, strong and dry soils, when sugar beet harvesting becomes associated with increased power consumption and the reduced durability of beet harvesting machinery.

In its turn, sound theoretical analysis of any work process (including a vibrational one) is possible only after developing appropriate mathematical models representing that process. Moreover, numerical modelling with the use of the developed mathematical models (numerical experiment) allows to reduce significantly the time and resources spent for the experimental study and full-scale testing of new units (Bulgakov, 2005; Bulgakov, 2011).

A fundamental theoretical and experimental study of the vibration-assisted sugar beet root lifting was presented in the work (Vasilenko et al., 1970). In that study the sugar beet root was modelled as a body with elastic properties approximated by a rod with a variable cross-section and one end fixed, which was under the effect of the perturbing force applied in the vertical and transverse plane. But the process of direct sugar beet root lifting from the soil was virtually left aside in that work, it only stated that, using additionally generated kinetostatic equations, it is possible to find out the terms of the complete lifting of a beet root from the soil.

The effectively first monograph on the theory of sugar beet harvesting machinery (Pogorely et el., 1983), regrettably, also does not examine theoretically the direct process of root lifting from the soil with the use of vibrating lifting units.

A significant amount of the results of scientific (mostly experimental) research into the lifting units of beet harvesters has been published for the recent years, but no results has appeared on the vibration-assisted root lifting.

The further development of the theory of vibration-assisted sugar beet root lifting can be found in the works (Bulgakov, 2005; Bulgakov et al., 2005; Bulgakov et al., 2014; Bulgakov et al., 2015a). For example, the paper (Bulgakov, 2005) formulates a new theory of the natural and forced longitudinal oscillations of the beet root body induced by the action of the vertical perturbing force. The said theory was developed in order to assess the effect the mentioned oscillations had on the process of breaking the bonds between the beet root and the soil and find the terms of maintaining the beet root intact during its lifting from the soil.

The same aim was pursued in the works (Bulgakov, 2005), which examined the transverse natural and forced oscillations of the beet root body occurring under the effect of a perturbing force acting along the line of motion of the lifting unit.

The paper (Bulgakov & Ivanovs, 2010) considers the process of lifting the beet root from the soil in the most general case – when the vibrational lifting unit grips the root non-symmetrically. The process is described using kinematic and dynamic Euler equations. The differential equation system obtained in the study characterizes the process of the three-dimensional oscillations of the root fixed in the soil, placed in elastic medium with one fixed point.

Meanwhile, the process of vibration-assisted beet root lifting from the soil is studied in the said work presuming the symmetrical gripping of the root by both vibrational lifting unit shares, since the non-symmetrical gripping of the root by one share goes on only for a short while. As a consequence of the translational movement of the vibrational lifting unit and the tapering of the working passage, the unit will further grip the root on both sides. But if the beet root is located on the vibrational lifting unit's symmetry axis, then the root is gripped on both sides straight from the beginning. And that is just the mode of beet root gripping by the digging shares, which enables the process of direct vibration-assisted lifting of the root from the soil.

The aim of this study is to develop the fundamentals of a theory of the direct vibration-assisted beet root lifting from the soil under the effect of the vertical perturbing force imparted to the root by the vibrational lifting unit and the pulling force generated by the unit's translational movement.

MATERIALS AND METHODS

To make an analytical description of the above-mentioned work process of beet root lifting from the soil it is necessary first to define the equivalent schematic representation and choose the required systems of coordinates (Vasilenko, 1996).

For that purpose, we represent the vibrational lifting unit by two wedges (digging shares or planes): $A_1B_1C_1$ and $A_2B_2C_2$, each of them being inclined at the angles α, β, γ and positioned relative to each other so as to form a working passage tapering rearwards (Fig. 1). The said wedges $A_1B_1C_1$ and $A_2B_2C_2$ oscillate in the longitudinal vertical plane (the digging share oscillation drive mechanism is not shown). The direction of the lifting unit's translational movement is shown by an arrow. The projections of the points B_1 and B_2 on the axis O_1y_1 are designated by the points D_1 and D_2 respectively.

It is assumed that the beet root approximated by a cone-shaped body interacts at the respective points with the surfaces of the wedges $A_1B_1C_1$ and $A_2B_2C_2$ and also the vibrating lifter grips the root on two sides. Further we suppose that the working surface of the wedge $A_1B_1C_1$ makes direct contact with the cone-shaped beet root body at the point K_1 the surface of the wedge $A_2B_2C_2$ – at the point K_2. Following that, the right lines drawn through the beet root contact points K_1 and K_2 and the points B_1 and B_2, when crossing the sides of the wedges A_1C_1 and A_2C_2, generate the corresponding points M_1 and M_2.

Hence, δ is the dihedral angle ($\angle B_1M_1D_1$) between the lower base $A_1D_1C_1$ and the wedge working surface $A_1B_1C_1$ or the same dihedral angle between the lower base $A_2D_2C_2$ and the second wedge working surface $A_2B_2C_2$.

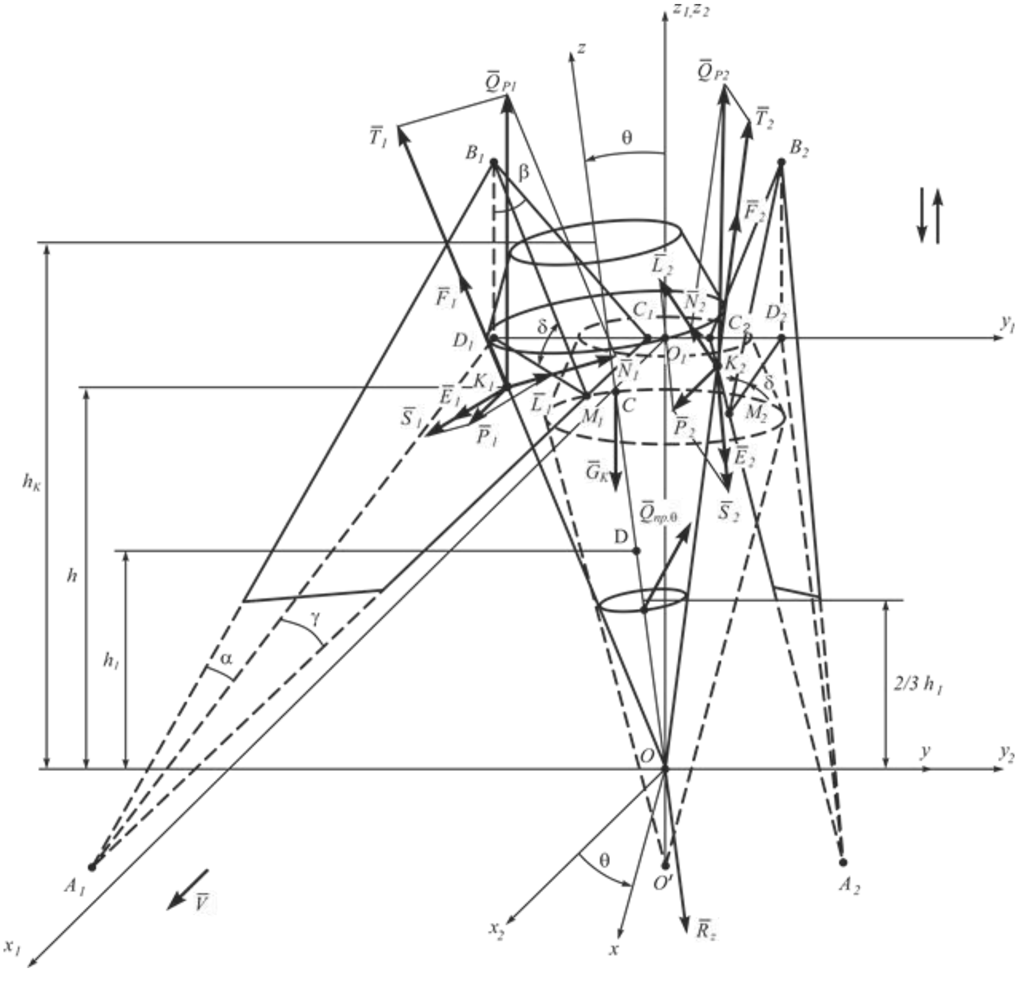


Figure 1. Equivalent schematic model of interaction between vibrational lifting unit and sugar beet root during its lifting from the soil.

Now we are going to associate with the vibrational lifting unit the orthogonal Cartesian coordinate system $x_1O_1y_1z_1$, the centre O_1 of which is placed in the middle of the unit's necked-in passage, the axis O_1x_1 is in line with the direction of the unit's translational movement, the axis O_1z_1 is vertically pointing up, and the axis O_1y_1 is pointing to the right.

Thus, the displacement of the beet root during its direct lifting from the soil shall be viewed with reference to the fixed system of coordinates $x_1O_1y_1z_1$. Further, we introduce the moving system of coordinates $x_cOy_cz_c$ rigidly bound to the beet root, its origin being placed at the root's centre of mass (point C), the axis Cz_c being in line with the beet root symmetry axis, the axes Cx_c and Cy_c lying in the plane that is perpendicular to the axis Cz_c.

Let’s examine the process of beet root lifting from the soil with the use of the vibration method in detail from the physical point of view. As the process of oscillation of the beet root itself in the soil as an elastic medium progresses, the bonds between the root and the soil break at a high rate and, accordingly, the restoring forces start sharply decreasing. Following that, the oscillatory process transforms into the process of continuous displacement of the beet root along O_1x_1 and O_1z_1 as well as the continuous angular displacement (turning) of the root around its centre of mass (point C) through some angle θ without the root returning to its initial position.

Accordingly, we arrive to the stage of direct lifting of the beet root from the soil. The process of transition from the beet root oscillatory motion to its continuous displacement in the soil can be described in more detail as follows. Under the effect of the vertical perturbing force the beet root performs translational oscillations together with the surrounding soil, and also the closer the soil is to the beet root, the more these soil oscillations are synchronized with the beet root oscillations. And vice versa – the further the soil is away from the beet root, the less its oscillations copy the beet root oscillations, due to the elastoplastic properties of the soil. Finally, there exists such a distance from the beet root, at which the soil does not oscillate at all, but the limits of the soil area that can oscillate together with the beet root are not outlined quite exactly (all depends on the soil’s mechanical-and-physical properties). It is most likely that smooth transition from the soil area oscillating together with the root to the area with no oscillation takes place, and therefore the breaking of the soil at the interface of these areas is unlikely. Supposedly, the most probable scenario is the soil breaking in immediate proximity to the root body surface or even on the beet root body surface itself. This provides the most believable explanation to the fact that during the vibration-assisted sugar beet root lifting a considerably smaller amount of soil remains stuck to the root’s sides than in case of the similar lifting with the use of conventional share (or disc) lifting units. As the lifting of the root from the soil can take place, as it was shown earlier, only in case the vibrational lifting unit grips the beet root symmetrically, so, simultaneously with the beet root’s translational oscillations, its oscillations through a certain angle about the conditional point of its fixation O take place.

At the first stage of the beet root lifting from the soil, especially during the first oscillations, the restoring force that acts during the angular oscillations and, of course, its moment about the fixation point O are maximal. Therefore, it is most likely that the beet root tilt angle will be insignificant and so full restoration of the root’s vertical position or partial restoration of such a position due to the translational movement of the vibrational lifting unit can be expected. But, under the effect of the translational oscillations of the beet root itself together with the surrounding soil the density of the surrounding soil will decrease, so, the restoring force during the angular oscillations will decrease as well. Thus, with each new oscillation the beet root tilt angle will increase, while its restoration to the initial position will decrease. In practice, the beet root will get looser and looser swinging about its conditional fixation point O with the gradual growth of the angle of its tilt forward along the lifting unit’s line of motion. This will promote the breaking of bonds between the beet root and the soil along the axis O_1x_1, starting from the upper part of its conical surface situated in unbroken soil and gradually proceeding towards the fixation point O. Thereby, the above-said implies that the breakup of bonds between the beet root and the soil occurs simultaneously in two directions: along the axes O_1x_1 and O_1z_1. In such a case, the forces binding the beet root with the soil and the elastic forces in the soil will be gradually decreasing until they reach such a minimum level that the oscillatory (vibration) processes will transform into the process of continuous displacement of the beet root upwards along the axis O_1z_1 and forward along the axis O_1x_1 and continuous rotation of the root body around the centre of mass (point C) through certain angle θ up to the complete lifting from the soil. Meanwhile, the elastic forces will just transform into the loosened soil resistance force during the beet root's movement in the working passage of the vibrational lifting unit.

In order to represent the described physical process of the beet root lifting from the soil we show in the equivalent schematic model the forces generated by the interaction between the beet root and the inside surface of the lifting unit working passage.

Now we assume that the vibrational lifting unit exerts, as it was indicated earlier, the vertical perturbing force $\bar{Q}_p$, which varies according to the following harmonic law:

$$Q_p = H\sin(\omega t), \tag{1}$$

where H – amplitude of perturbing force; ω – frequency of perturbing force; t – time.

This force plays the main role in the process of loosening the soil in the vibrational lifting unit working passage area and lifting the beet root from the soil.

The denoted perturbing force $\bar{Q}_p$ is applied to the beet root on both sides of it, therefore, it is represented in the equivalent schematic model by its two components $\bar{Q}_{p1}$ and $\bar{Q}_{p2}$. These forces are applied at the points K_1 and K_2, respectively, at a distance of h from the conditional fixation point O and they are exactly the forces inducing the beet root's oscillations in the longitudinal and vertical plane as well as breaking the bonds between the beet root and the soil and providing the conditions needed for lifting the root completely from the soil.

Since the beet root gripping is symmetrical, obviously, the following relation is going to be observed:

$$Q_{p1} = Q_{p2} = \frac{1}{2} H \sin(\omega t). \tag{2}$$

We resolve these perturbing forces into the normal components $\bar{N}_1$ and $\bar{N}_2$ and tangential components $\bar{T}_1$ and $\bar{T}_2$, as shown in Fig. 1. The compositions of the forces will be as follows:

$$\bar{Q}_{p1} = \bar{N}_1 + \bar{T}_1, \tag{3}$$

$$\bar{Q}_{p2} = \bar{N}_2 + \bar{T}_2. \tag{4}$$

Apparently, the lines of the force vectors $\bar{T}_1$ and $\bar{T}_2$ will be parallel to the right lines B_1M_1 and B_2M_2, respectively.

As the vibrational lifting unit advances linearly along the axis O_1x_1 with respect to the beet root fixed in the soil, so at the moment, when the unit grips the root, there are also moving forces $\overline{P}_1$ and $\overline{P}_2$ acting along the axis O_1x_1. As we did earlier, we resolve the moving forces $\overline{P}_1$ and $\overline{P}_2$ into the normal components $\overline{L}_1$ and $\overline{L}_2$ and tangential components $\overline{S}_1$ and $\overline{S}_2$ with reference to the planes $A_1B_1C_1$ and $A_2B_2C_2$, respectively, i.e.:

$$\overline{P}_1 = \overline{L}_1 + \overline{S}_1, \tag{5}$$

$$\overline{P}_2 = \overline{L}_2 + \overline{S}_2. \tag{6}$$

The force vectors $\overline{S}_1$ and $\overline{S}_2$ act along the vector lines of speed of the shares relative to the root surface during the translational motion of the vibrational lifting unit.

Thus, the lifting wedges $A_1B_1C_1$ and $A_2B_2C_2$ exert on the sugar beet root the following forces at the contact points K_1 and K_2:

$$\overline{N}_{K1} = \overline{N}_1 + \overline{L}_1, \tag{7}$$

$$\overline{N}_{K2} = \overline{N}_2 + \overline{L}_2, \tag{8}$$

which act along the normal to the planes $A_1B_1C_1$ and $A_2B_2C_2$, respectively.

Obviously, the magnitudes of these forces are as follows:

$$N_{K1} = N_1 + L_1, \tag{9}$$

$$N_{K2} = N_2 + L_2. \tag{10}$$

Besides that, at the contact points K_1 and K_2 the friction forces $\overline{F}_{K1}$ and $\overline{F}_{K2}$, respectively, are applied, which counteract the slipping of the beet root body on the working surfaces of the wedges $A_1B_1C_1$ and $A_2B_2C_2$, when the lifting unit grips the root. The vectors of these forces have the directions that are opposite to the directions of the vectors of the relative speed of the beet root slipping on the surfaces of the said wedges.

At the root's centre of gravity (point C), the root weight force $\overline{G}_k$ is applied. The forces of resistance exerted by the loosened soil during the beet root's movement in the working passage of the vibrational lifting unit along the axes O_1x_1 and O_1z_1 are designated $\overline{R}_{x1}$ and $\overline{R}_{z1}$, respectively.

During the direct beet root lifting from the soil, the rotation of the root around its centre of mass C takes place under the effect of the couple of resistance forces exerted by the loosened soil. The moment of the couple of forces will be designated *M*.

Now we are going to find the magnitudes of the forces shown in Fig. 1. The tangential components $\overline{T}_1$ and $\overline{T}_2$ of the perturbing forces $\overline{Q}_{p1}$ and $\overline{Q}_{p2}$, respectively, and the tangential components $\overline{S}_1$ and $\overline{S}_2$ of the moving forces $\overline{P}_1$ and $\overline{P}_2$, respectively, do not have any direct effect on the beet root, they only produce loosening of the soil around the root.

It should be noted, taking into account the symmetry of the beet root gripping by the vibrational lifting unit, that the same name forces generated on the two share working surfaces during their interaction with the beet root will have equal magnitudes and symmetrical lines of action with respect to the symmetry plane $x_1O_1z_1$ (Fig. 1). Accordingly, from the schematic model of forces we derive the formulae for finding the normal components $\overline{N}_1$ and $\overline{N}_2$ and the tangential components $\overline{T}_1$ and $\overline{T}_2$ of the perturbing forces $\overline{Q}_{p1}$ and $\overline{Q}_{p2}$. They have the following values:

$$N_1 = N_2 = Q_{p1} \cos \delta, \tag{11}$$

$$T_1 = T_2 = Q_{p1} \sin \delta. \tag{12}$$

From the same schematic force model the formulae for finding the normal components $\overline{L}_1$ and $\overline{L}_2$ and tangential components $\overline{S}_1$ and $\overline{S}_2$ of the moving forces $\overline{P}_1$ and $\overline{P}_2$, respectively, can be derived:

$$L_1 = L_2 = P_1 \sin \gamma , \tag{13}$$

$$S_1 = S_2 = P_1 \cos \gamma . \tag{14}$$

The magnitudes of the forces $\overline{N}_{K1}$ and $\overline{N}_{K2}$ are as follows, taking into account the expressions (9), (11) and (13):

$$N_{K1} = N_{K2} = Q_{\mathrm{p}1} \cos \delta + P_1 \sin \gamma , \tag{15}$$

or, considering the expression (2), we come to the following:

$$N_{K1} = N_{K2} = \frac{1}{2} H \cos \delta \sin(\omega t) + P_1 \sin \gamma . \tag{16}$$

Hence, the magnitudes of the friction forces $\overline{F}_{K1}$ and $\overline{F}_{K2}$ are:

$$F_{K1} = F_{K2} = f N_{K1} = f\left(Q_{p1} \cos \delta + P_1 \sin \gamma\right), \tag{17}$$

or, considering the expression (2), we come to:

$$F_{K1} = F_{K2} = \frac{1}{2} f H \cos \delta \sin(\omega t) + f P_1 \sin \gamma , \tag{18}$$

where *f* – coefficient of friction.

Apparently, during the immediate contact between the wedges $A_1B_1C_1$ and $A_2B_2C_2$ and the beet root surface, the friction force vectors $\bar{F}_{K1}$ and $\bar{F}_{K2}$ always lie in the wedge planes $A_1B_1C_1$ and $A_2B_2C_2$, respectively. Besides that, due to the soil resistance forces, the slipping of the beet root on the surfaces of the wedges along the lines of action of the forces $\bar{T}_1$, $\bar{T}_2$ (parallel to the lines B_1M_1 and B_2M_2) and in the direction opposite to the forces $\bar{S}_1$ and $\bar{S}_2$ is possible.

Therefore, the vector of the relative speed of the beet root slipping on the surfaces of the wedges can be resolved into components in the above said directions. Thus, the friction force $\bar{F}_{K1}$ can also be resolved into two components: $\bar{F}_1$ – in the direction opposite to the vector $\bar{T}_1$, and $\bar{E}_1$ – in the direction of the vector $\bar{S}_1$, i.e.:

$$\bar{F}_{K1} = \bar{F}_1 + \bar{E}_1 . \tag{19}$$

Similarly, the friction force $\bar{F}_{K2}$ can as well be resolved into two components: $\bar{F}_2$ – in the direction opposite to the vector $\bar{T}_2$, and $\bar{E}_2$ – in the direction of the vector $\bar{S}_2$, i.e.:

$$\bar{F}_{K_2} = \bar{F}_2 + \bar{E}_2 . \tag{20}$$

Obviously, $F_1 = F_2$, $E_1 = E_2$.

Now let’s find the magnitudes of the components of the forces $\bar{F}_1$ and $\bar{E}_1$, and consequently $\bar{F}_2$ and $\bar{E}_2$. Basing on the above considerations and expression (16), a deduction can be drawn that in the intervals $[2k\pi, (2k + 1)\pi]$, $k = 0, 1, 2, \ldots$, particularly in the interval $[0, \pi]$, the magnitude of the friction force $\bar{F}_{K1}$ ($\bar{F}_{K2}$) shall be determined in accordance with the formula (18), moreover, in the interval $\left[0, \frac{\pi}{2}\right]$ it rises from its minimum value:

$$F_{K1\min} = F_{K2\min} = f P_1 \sin\gamma , \tag{21}$$

to the maximum value:

$$F_{K1\max} = F_{K2\max} = \frac{1}{2} f H \cos\delta + f P_1 \sin\gamma , \tag{22}$$

While in the interval $\left[\frac{\pi}{2}, \pi\right]$ it decreases from $\bar{F}_{K1\max}$ ($\bar{F}_{K2\max}$) to $\bar{F}_{K1\min}$ ($\bar{F}_{K2\min}$). Besides that, the direction of the friction force vector in the interval $\left[0, \frac{\pi}{2}\right]$ also changes. The vector $\bar{F}_{K1\min}$ ($\bar{F}_{K2\min}$) has the same direction as the friction force vector of a usual share lifter (in the absence of any perturbing force), i.e. parallel to the right lines A_1O_1' (A_2O_2'), while $\angle O_1'A_1M_1 = \angle O_2'A_2M_2 = \gamma$ (Bulgakov, 2005). The vector $\bar{F}_{K1\max}$ ($\bar{F}_{K2\max}$) deflects from the vector $\bar{F}_{K1\min}$ ($\bar{F}_{K2\min}$) through a certain angle $\alpha_{K1max}(\alpha_{K2max})$, while $\alpha_{K1max} = \alpha_{K2max}$.

So, in the interval $\left[0, \frac{\pi}{2}\right]$ the force vector $\bar{F}_{K1}$ ($\bar{F}_{K2}$) changes from the vector $\bar{F}_{K1\min}$ ($\bar{F}_{K2\min}$) to the vector $\bar{F}_{K1\max}$ ($\bar{F}_{K2\max}$), and in the interval $\left[\frac{\pi}{2}, \pi\right]$ – from the vector $\bar{F}_{K1\max}$ ($\bar{F}_{K2\max}$) to the vector $\bar{F}_{K1\min}$ ($\bar{F}_{K2\min}$). Hence, the angle $\alpha_{K1}(\alpha_{K2})$ of the deflection of the vector $\bar{F}_{K1}$ ($\bar{F}_{K2}$) from the vector $\bar{F}_{K1\min}$ ($\bar{F}_{K2\min}$) changes in the interval $[0, \pi]$ under the following law:

$$\alpha_{K2} = \alpha_{K1} = \alpha_{K1\max} \sin(\omega t) . \tag{23}$$

Apparently, the value $\alpha_{K1\max}$ ($\alpha_{K2\max}$) depends first of all on the ratio $\frac{H}{P_1}$ $\left(\frac{H}{P_2}\right)$ and the greater the ratio is, the greater the value grows. Therefore, in the interval [0, π] the magnitude of the friction force vector $\bar{F}_{K1}$ ($\bar{F}_{K2}$) changes according to the law (18), while its direction – according to the law (23).

So, in the interval [0, π] we have the following values of the component forces $\bar{F}_1$ ($\bar{F}_2$) and $\bar{E}_1$($\bar{E}_2$):

$$F_1 = F_2 = F_{K1} \sin(\gamma + \alpha_{K1}), \tag{24}$$

$$E_1 = E_2 = F_{K1} \cos(\gamma + \alpha_{K1}), \tag{25}$$

then, taking into account (18) and (23), we obtain:

$$F_1 = F_2 = \left(\frac{1}{2} f H \cos\delta \sin(\omega t) + f P_1 \sin\gamma\right) \sin\left(\gamma + \alpha_{K1\max} \sin(\omega t)\right), \tag{26}$$

$$E_1 = E_2 = \left(\frac{1}{2} f H \cos\delta \sin(\omega t) + f P_1 \sin\gamma\right) \cos\left(\gamma + \alpha_{K1\max} \sin(\omega t)\right). \tag{27}$$

The formulae (26) and (27) are effective in any of the intervals [2kπ, (2k + 1)π], k = 0, 1, 2, …

Obviously, within the intervals [(2k + 1)π, 2kπ], k = 0, 1, 2, …, the friction forces $\bar{F}_{K1}$ ($\bar{F}_{K2}$) are as follows:

$$F_{K1} = F_{K2} = F_{K1\min} = f P_1 \sin\gamma . \tag{28}$$

Hence, the following is observed in the denoted intervals:

$$F_1 = F_2 = F_{K1\min} \sin\gamma = f P_1 \sin\gamma \sin\gamma = f P_1 \sin^2\gamma , \tag{29}$$

$$E_1 = E_2 = F_{K1\min} \cos\gamma = f P_1 \sin\gamma \cos\gamma = \frac{1}{2} f P_1 \sin 2\gamma . \tag{30}$$

It can be assumed that the loosened soil resistance forces during the direct beet root lifting from the soil are a function of the speed, with which the beet root travels in the loosened soil. However, as a first approximation, the magnitudes of these forces can be regarded constant. Hence, overall, in order to simplify the mathematical model, we consider the forces $\bar{R}_{x1}$, $\bar{R}_{z1}$ and the moment of couple M to be constant.

Taking into account the equivalent schematic model (model of forces) drawn above, the differential equations of motion of the beet root's centre of mass during its direct lifting from the soil in the vectorial form will be as follows:

$$m_k\overline{a} = \overline{N}_1 + \overline{N}_2 + \overline{L}_1 + \overline{L}_2 + \overline{F}_1 + \overline{F}_2 + \overline{E}_1 + \overline{E}_2 + \overline{G}_k + \overline{R}_{z1} + \overline{R}_{x1}, \qquad (31)$$

where m_k is the beet root mass; $\overline{a}$ is the acceleration of the beet root's centre of mass.

Further, we are going to derive the differential equations of motion of the beet root's centre of mass (point C) during its translational movement along the axes O_1x_1 and O_1z_1. As the process of beet root lifting from the soil occurs in case of the lifting unit symmetrically gripping the root body, so the root's movement along the working passage effectively takes place in the longitudinal and vertical plane (plane $x_1O_1z_1$), therefore, the vector equation (31) is reduced to differential equations in the projections on the axes Ox_1 and Oz_1 of the following form:

$$\left.\begin{aligned} m_k\ddot{x}_1 &= N_{1x1} + N_{2x1} + L_{1x1} + L_{2x1} + F_{1x1} + F_{2x1} + E_{1x1} + E_{2x1} - R_{x1}, \\ m_k\ddot{z}_1 &= N_{1z1} + N_{2z1} + L_{1z1} + L_{2z1} - F_{1z1} - F_{2z1} - G_k - R_{z1}. \end{aligned}\right\} \qquad (32)$$

Let's determine the values of the force projections on the axes Ox_1 and Oz_1 used in the system of equations (32). Taking into account the formulae derived in (Bulgakov et al., 2015b), the projections of the normal components $\overline{N}_1$ and $\overline{N}_2$ on the axis O_1x_1 are determined as follows:

$$\overline{N}_{1x1} = \overline{N}_{2x1} = \frac{N_1 \tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}}, \qquad (33)$$

or, taking into account the expression (11), we obtain:

$$\overline{N}_{1x1} = \overline{N}_{2x1} = \frac{Q_{p1}\cos\delta\tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}}. \qquad (34)$$

The projections of the normal components $\overline{L}_1$ and $\overline{L}_2$ on the axis O_1x_1 have the following values:

$$L_{1x1} = L_{2x1} = \frac{L_1 \tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}}, \qquad (35)$$

or, taking into account the expression (13), we obtain:

$$L_{1x1} = L_{2x1} = \frac{P_1 \sin\gamma\tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}}. \qquad (36)$$

For the projections of the friction force components $\bar{F}_1$ and $\bar{F}_2$ the following expressions are obtained:

$$F_{1x1} = F_{2x1} = F_1 \cos\delta \sin\gamma\,, \tag{37}$$

or, taking into account the expression (26), we have:

$$F_{1x1} = F_{2x1} = \left[\frac{1}{2} f H \cos\delta \sin(\omega t) + f P_1 \sin\gamma\right] \times \\ \times \sin[\gamma + \alpha_{K1max} \sin(\omega t)] \cos\delta \sin\gamma, \tag{38}$$

$$\omega t \in [2k\pi, (2k+1)\pi], k = 0, 1, 2,...$$

Taking into consideration the formula (29), we come to:

$$F_{1x1} = F_{2x1} = f P_1 \sin^3\gamma \cos\delta, \\ \omega t \in \left[(2k-1)\pi, \quad 2k\pi\right], \qquad k = 1, 2,... \tag{39}$$

The projections of the friction force components $\bar{E}_1$ and $\bar{E}_2$ on the axis O_1x_1 will be as follows:

$$E_{1x1} = E_{2x1} = E_1 \cos\gamma\,, \tag{40}$$

or, taking into account (27), we have the following expression:

$$E_{1x1} = E_{2x1} = \left[\frac{1}{2} f H \cos\delta \sin(\omega t) + f P_1 \sin\gamma\right] \\ \times \cos[\gamma + \alpha_{K1max} \sin(\omega t)] \cos\gamma, \tag{41}$$

$$\omega t \in [2k\pi, (2k+1)\pi], k = 0, 1, 2,...$$

Taking into account the expression (30), we obtain:

$$E_{1x1} = E_{2x1} = \frac{1}{2} f P_1 \sin 2\gamma \cos\gamma, \\ \omega t \in \left[(2k-1)\pi, \quad 2k\pi\right], \qquad k = 1, 2,... \tag{42}$$

The projections of the friction force components $\bar{E}_1$ and $\bar{E}_2$ on the axis O_1z_1 are equal to zero in any interval, i.e. $E_{1z1} = E_{2z1} = 0$.

The projections of the normal components $\bar{N}_1$ and $\bar{N}_2$ on the axis O_1z_1, according to (Bulgakov et al., 2015b), are as follows:

$$N_{1z1} = N_{2z1} = \frac{N_1 \tan\beta}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}}\,, \tag{43}$$

or, taking into account the expression (11), we come to:

$$N_{1z1} = N_{2z1} = \frac{Q_{p1} \cos\delta \tan\beta}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} . \tag{44}$$

The projections of the normal components $\bar{L}_1$ and $\bar{L}_2$ on the axis O_1z_1 will be equal to:

$$L_{1z1} = L_{2z1} = \frac{L_1 \tan\beta}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} , \tag{45}$$

or, taking into account the expression (13), we have:

$$L_{1z1} = L_{2z1} = \frac{P_1 \sin\gamma \tan\beta}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} . \tag{46}$$

The projections of the friction force components $\bar{F}_1$ and $\bar{F}_2$ on the axis O_1z_1 will be equal to:

$$F_{1z1} = F_{2z1} = F_1 \sin\delta , \tag{47}$$

or, taking into account the expression (26), we have:

$$\begin{gathered} F_{1z1} = F_{2z1} = \left[\frac{1}{2} fH \cos\delta \sin(\omega t) + fP_1 \sin\gamma\right] \\ \times \sin[\gamma + \alpha_{K1max} \sin(\omega t)] \sin\delta , \\ \omega t \in [2k\pi, (2k+1)\pi], k = 0, 1, 2,... \end{gathered} \tag{48}$$

Taking into consideration the formula (29), we obtain:

$$\begin{gathered} F_{1z1} = F_{2z1} = f P_1 \sin^2\gamma \sin\delta, \\ \omega t \in \left[(2k-1)\pi,\ \ 2k\pi\right], \quad k = 1, 2,... \end{gathered} \tag{49}$$

By substituting the expressions (34), (36), (38) or (39), (41) or (42), (44), (46), (48) or (49) into the system of differential equations (32), we obtain the following system of differential equations:

$$\left.\begin{aligned}
m_k\ddot{x}_1 = {} & \frac{2Q_{p1}\cos\delta\tan\gamma}{\sqrt{(\tan\gamma)^2+1+(\tan\beta)^2}} + \frac{2P_1\sin\gamma\tan\gamma}{\sqrt{(\tan\gamma)^2+1+(\tan\beta)^2}} \\
& + +2\left[\frac{1}{2}fH\cos\delta\sin(\omega t) \right. \\
& \left. + fP_1\sin\gamma\right]\sin[\gamma+\alpha_{K1max}\sin(\omega t)]\cos\delta\sin\gamma \\
& + 2\left[\frac{1}{2}fH\cos\delta\sin(\omega t) \right. \\
& \left. + fP_1\sin\gamma\right]\cos[\gamma+\alpha_{K1max}\sin(\omega t)]\cos\gamma - R_{x1} \\
m_k\ddot{z}_1 = {} & \frac{2Q_{p1}\cos\delta\tan\beta}{\sqrt{(\tan\gamma)^2+1+(\tan\beta)^2}} + \frac{2P_1\sin\gamma\tan\beta}{\sqrt{(\tan\gamma)^2+1+(\tan\beta)^2}} \\
& - 2\left[\frac{1}{2}fH\cos\delta\sin(\omega t) \right. \\
& \left. + fP_1\sin\gamma\right]\sin[\gamma+\alpha_{K1max}\sin(\omega t)]\sin\delta - G_k - R_{z1}, \\
& \omega t \in [2k\pi,(2k+1)\pi], \qquad k=1,2,\ldots
\end{aligned}\right\} \quad (50)$$

or:

$$\left.\begin{aligned}
m_k\ddot{x}_1 = {} & \frac{2P_1\sin\gamma\tan\gamma}{\sqrt{\tan^2\gamma+1+\tan^2\beta}} \\
& + 2fP_1\sin^3\gamma\cos\delta + fP_1\sin 2\gamma\cos\gamma - R_{x1}, \\
m_k\ddot{z}_1 = {} & \frac{2P_1\sin\gamma\tan\beta}{\sqrt{\tan^2\gamma+1+\tan^2\beta}} - 2fP_1\sin^2\gamma\sin\delta - G_k - R_{z1}, \\
& \omega t \in [(2k-1)\pi, 2k\pi],\ k=1,2,\ldots
\end{aligned}\right\} \quad (51)$$

In the systems of equations (50), (51) the magnitudes of the loosened soil resistance forces $\bar{R}_{x1}$ and $\bar{R}_{z1}$ acting during the beet root's movement in the working passage of the vibrational lifting unit are regarded constant.

Now we are going to establish the initial conditions for the differential equations (50), (51). Since the beet root prior to the start of its direct lifting from the soil performs oscillations about the equilibrium position, the following can be taken as the initial conditions for the coordinates of the root's centre of mass (point C):

at $t = 0$:

$$x_1 = x_{10}, \qquad z_1 = -\frac{1}{3}h_k,$$

Where x_{10} is the distance from the vertical centreline of the beet root to the origin of coordinates (point O_1) at the time point $t = 0$.

An error, if any, can arise only within the limits of the beet root oscillation amplitude, which is very insignificant as compared with the length of the lifting unit working passage and the running depth in soil, where the root lifting is done. Considering further that during each oscillation, within the whole period, the instants exist, when the beet root displacement velocity is equal to zero, we take as the initial time point exactly such an instance during the last oscillation followed further by direct beet root lifting from the soil.

Thus, the initial conditions for the systems of differential equations (50), (51) will be as follows:
at $t = 0$:

$$\dot{x}_1 = 0, \qquad \dot{z}_1 = 0, \qquad x_1 = x_{10}, \qquad z_1 = -\frac{1}{3}h_k. \tag{52}$$

After substituting the expression (2) into the system of equations (50) and making certain transformations, we obtain the following system of differential equations:

$$\left.\begin{aligned}
\ddot{x}_1 = {} & \frac{1}{m_k}\left[\frac{\cos\delta\tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}}\right. \\
& + f\cos^2\delta\sin[\gamma + \alpha_{K1max}\sin(\omega t)]\sin\gamma \\
& \left. + f\cos\delta\cos[\gamma + \alpha_{K1max}\sin(\omega t)]\cos\gamma\right] \\
& \times H\sin(\omega t) \\
& + \frac{2}{m_k}\left[\frac{\sin\gamma\tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}}\right. \\
& + f\sin^2\gamma\sin[\gamma + \alpha_{K1max}\sin(\omega t)]\cos\delta \\
& \left. + f\sin\gamma\cos\gamma\cos[\gamma + \alpha_{K1max}\sin(\omega t)]\right] P_1 - \frac{R_{x1}}{m_k}, \\
\ddot{z}_1 = {} & \frac{1}{m_k}\left[\frac{\cos\delta\tan\beta}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}}\right. \\
& \left. - f\cos\delta\sin[\gamma + \alpha_{K1max}\sin(\omega t)]\sin\delta\right] \\
& \times H\sin(\omega t) + \frac{2}{m_k}\left[\frac{\sin\gamma\tan\beta}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}}\right. \\
& \left. - f\sin\gamma\sin[\gamma + \alpha_{K1max}\sin(\omega t)]\sin\delta\right] P_1 - \frac{G_k}{m_k} \\
& - \frac{R_{z1}}{m_k}, \\
& \omega t \in [2k\pi, (2k+1)\pi], \qquad k = 1, 2, \ldots
\end{aligned}\right\} \tag{53}$$

The system of differential equations (53) is nonlinear. It can be integrated only with the use of approximate numerical methods on a PC. First, we are going to make certain assumptions. As a first approximation, we assume that the friction force vectors $\bar{F}_{K1}$ and

$\bar{F}_{K2}$ maintain a constant direction, i.e. the angle between the vectors $\bar{F}_{K1\min}$ and $\bar{F}_{K1}$ is constant and equal to $\frac{\alpha_{K1max}}{2}$, similarly, the angle between the vectors $\bar{F}_{K2\min}$ and $\bar{F}_{K2}$ is also constant and equal to $\frac{\alpha_{K2max}}{2}$, while $\frac{\alpha_{K2max}}{2} = \frac{\alpha_{K1max}}{2}$.

Taking into account these assumptions, the system of differential equations (53) acquires the following form:

$$\left.\begin{aligned}
\ddot{x}_1 = {} & \frac{1}{m_k}\left[\frac{\cos\delta\tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} \right. \\
& + f\cos^2\delta\sin\left(\gamma + \frac{\alpha_{K1max}}{2}\right)\sin\gamma \\
& \left. + f\cos\delta\cos\left(\gamma + \frac{\alpha_{K1max}}{2}\right)\cos\gamma\right] H\sin(\omega t) \\
& + \frac{2}{m_k}\left[\frac{\sin\gamma\tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} \right. \\
& + f\sin^2\gamma\sin\left(\gamma + \frac{\alpha_{K1max}}{2}\right)\cos\delta \\
& \left. + f\sin\gamma\cos\gamma\ \cos\left(\gamma + \frac{\alpha_{K1max}}{2}\right)\right] P_1 - \frac{R_{x1}}{m_k}, \\
\ddot{z}_1 = {} & \frac{1}{m_k}\left[\frac{\cos\delta\tan\beta}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} - f\cos\delta\sin\left(\gamma + \frac{\alpha_{K1max}}{2}\right)\sin\delta\right] \\
& \times H\sin(\omega t) + \frac{2}{m_k}\left[\frac{\sin\gamma\tan\beta}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} \right. \\
& \left. - f\sin\gamma\sin\left(\gamma + \frac{\alpha_{K1max}}{2}\right)\sin\delta\right] P_1 - \frac{R_{z1}}{m_k} - g\ , \\
& \omega t \in [2k\pi, (2k+1)\pi], \qquad k = 0, 1, 2, \ldots
\end{aligned}\right\} \tag{54}$$

where g – gravitational acceleration.

The system of differential equations (54) is a system of linear second-order differential equations. It can be solved by using the integration method.

To reduce the expression of the system of differential equations (54), we introduce the following designations.

$$\begin{aligned}
& \frac{1}{m_k}\left[\frac{\cos\delta\tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} \right. \\
& + f\cos^2\delta\sin\left(\gamma + \frac{\alpha_{K1max}}{2}\right)\sin\gamma \\
& \left. + f\cos\delta\cos\left(\gamma + \frac{\alpha_{K1max}}{2}\right)\cos\gamma\right] = \varphi_1\ ,
\end{aligned} \tag{55}$$

$$\left[\frac{\sin\gamma\tan\gamma}{\sqrt{\tan^2\gamma+1+\tan^2\beta}} + f\sin^2\gamma\sin\left(\gamma+\frac{\alpha_{K1max}}{2}\right)\cos\delta + f\sin\gamma\cos\gamma\cos\left(\gamma+\frac{\alpha_{K1max}}{2}\right)\right] = \psi_1 \,, \tag{56}$$

$$\frac{1}{m_k}\left[\frac{\cos\delta\tan\beta}{\sqrt{\tan^2\gamma+1+\tan^2\beta}} - f\cos\delta\sin\left(\gamma+\frac{\alpha_{K1max}}{2}\right)\sin\delta\right] = \varphi_2 \,, \tag{57}$$

$$\frac{2}{m_k}\left[\frac{\sin\gamma\tan\beta}{\sqrt{\tan^2\gamma+1+\tan^2\beta}} - f\sin\gamma\sin\left(\gamma+\frac{\alpha_{K1max}}{2}\right)\sin\delta\right] = \psi_2 \tag{58}$$

Taking into consideration the expressions (55) – (58), the system of differential equations (54) assumes the following form:

$$\left.\begin{aligned} \ddot{x}_1 &= \varphi_1 H\sin(\omega t) + \psi_1 P_1 - \frac{R_{x1}}{m_k}\,, \\ \ddot{z}_1 &= \varphi_2 H\sin(\omega t) + \psi_2 P_1 - \frac{R_{z1}}{m_k} - g\,. \end{aligned}\right\} \tag{59}$$

Now we are going to integrate the system of differential equations (59). The first integral will be as follows:

$$\left.\begin{aligned} \dot{x}_1 &= -\frac{\varphi_1}{\omega}\cos(\omega t) + \psi_1 P_1 t - \frac{R_{x1}}{m_k}t + C_1\,, \\ \dot{z}_1 &= -\frac{\varphi_2}{\omega}\cos(\omega t) + \psi_2 P_1 t - \frac{R_{z1}}{m_k}t - gt + L_1\,, \end{aligned}\right\} \tag{60}$$

where C_1 and L_1 are arbitrary constants.

The second integral of the system of differential equations (59) will be as follows:

$$\left.\begin{aligned} x_1 &= -\frac{\varphi_1 H}{\omega^2}\sin(\omega t) + \frac{\psi_1 P_1 t^2}{2} - \frac{R_{x1}t^2}{2m_k} + C_1 t + C_2\,, \\ z_1 &= -\frac{\varphi_2 H}{\omega^2}\sin(\omega t) + \frac{\psi_2 P_1 t^2}{2} - \frac{R_{z1}t^2}{2m_k} - \frac{gt^2}{2} + L_1 t + L_2\,, \end{aligned}\right\} \tag{61}$$

where C_2 and L_2 are arbitrary constants.

The arbitrary constants C_1, L_1, C_2 and L_2 are determined by the initial conditions (52). These arbitrary constants are equal to:

$$C_1 = \frac{\varphi_1 H}{\omega}, \qquad L_1 = \frac{\varphi_2 H}{\omega}, \qquad C_2 = x_{10}, \qquad L_2 = -\frac{1}{3} h_k \cdot \tag{62}$$

By substituting the values of the arbitrary constants C_1 and L_1 into the system of differential equations (60), we obtain:

$$\left.\begin{aligned} \dot{x}_1 &= -\frac{\varphi_1 H}{\omega}\cos(\omega t) + \psi_1 P_1 t - \frac{R_{x1} t}{m_k} + \frac{\varphi_1 H}{\omega}, \\ \dot{z}_1 &= -\frac{\varphi_2 H}{\omega}\cos(\omega t) + \psi_2 P_1 t - \frac{R_{z1} t}{m_k} + \frac{\varphi_2 H}{\omega}. \end{aligned}\right\} \tag{63}$$

By substituting the values of the derived arbitrary constants C_1, C_2, L_1 and L_2 into the system of equations (61), we obtain:

$$\left.\begin{aligned} x_1 &= -\frac{\varphi_1 H}{\omega^2}\sin(\omega t) + \frac{\psi_1 P_1 t^2}{2} - \frac{R_{x1} t^2}{2m_k} + \frac{\varphi_1 H t}{\omega} + x_{10}, \\ z_1 &= -\frac{\varphi_2 H}{\omega^2}\sin(\omega t) + \frac{\psi_2 P_1 t^2}{2} - \frac{R_{z1} t^2}{2m_k} - \frac{g t^2}{2} + \frac{\varphi_2 H t}{\omega} - \frac{1}{3} h_k. \end{aligned}\right\} \tag{64}$$

The systems of equations (63) and (64), respectively, characterize the laws of variation of the speed and displacement of the beet root's centre of mass in the process of its direct lifting from the soil. From the second equation of the system (64) the time t_1 of the direct beet root lifting from the soil can be found. For that purpose we have to substitute the value $z_1 = 0$ into the left member of the said equation and solve the resulting equation for t_1. Since the equation is transcendental, it is impossible to derive any analytic expression for finding t_1. However, it can be solved with the use of a PC applying the known methods. The computed value of t_1 can be subsequently used for determining the productivity of the sugar beet root harvesting machine equipped with vibrational lifting units.

Next we are going to give consideration to the system of differential equations (51). To reduce the expression of this system of equations, we again introduce the following designations:

$$\frac{1}{m_k}\left[\frac{2\sin\gamma\tan\gamma}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} + 2f\sin^3\gamma\cos\delta + f\sin(2\gamma)\cos\gamma\right] = \psi'_1 \tag{65}$$

$$\frac{1}{m_k}\left[\frac{2\sin\gamma\tan\beta}{\sqrt{\tan^2\gamma + 1 + \tan^2\beta}} - 2f\sin^2\gamma\sin\gamma\sin\delta\right] = \psi'_2 \tag{66}$$

Taking into account the expressions (65), (66), the system of differential equations (51) will take the following form:

$$\begin{aligned} M_{yc}(\bar{F_1}) = M_{yc}(\bar{F}_2) \\ &= \left(\frac{1}{2} f H \cos\delta \sin(\omega t) + f\, P_1 \sin\gamma\right) \\ &\times \sin[\gamma + \alpha_{K1max}\sin(\omega t)]\cos\gamma_k(-h_k + h - z_1)\sin\theta\,, \\ &\omega t \in [2k\pi, (2k+1)\pi], \qquad k = 0, 1, 2, \ldots \end{aligned} \tag{67}$$

After the first integration of the system of differential equations (67), we obtain:

$$\left.\begin{aligned} \dot{x}_1 &= \psi_1' P_1 t - \frac{R_{x_1}}{m_k} t + C_1, \\ \dot{z}_1 &= \psi_2' P_1 t - \frac{G_k}{m_k} t - \frac{R_{z_1}}{m_k} t + L_1, \end{aligned}\right\} \tag{68}$$

where C_1 and L_1 are arbitrary constants,

$$\omega t \in \left[(2k-1)\pi, \quad 2k\pi\right],\ k = 1, 2, \ldots$$

After the second integration of the system of differential equations (67), we obtain:

$$\left.\begin{aligned} x_1 &= \psi_1' P_1 \frac{t^2}{2} - \frac{R_{x_1} t^2}{2m_k} + C_1 t + C_2, \\ z_1 &= \psi_2' P_1 \frac{t^2}{2} - \frac{G_k t^2}{2m_k} - \frac{R_{z_1} t^2}{2m_k} + L_1 t + L_2, \end{aligned}\right\} \tag{69}$$

where C_2 and L_2 are arbitrary constants,

$$\omega t \in \left[(2k-1)\pi, \quad 2k\pi\right],\ k = 1, 2, \ldots$$

The arbitrary constants C_1, L_1, C_2 and L_2 are determined by the initial conditions (52). These arbitrary constants are equal to:

$$C_1 = 0, \qquad L_1 = 0, \qquad C_2 = x_{10}, \qquad L_2 = -\frac{1}{3} h_k\,. \tag{70}$$

By substituting the values of the arbitrary constants C_1 and L_1 into the system of equations (68), we obtain:

$$\left.\begin{aligned}\dot{x}_1 &= \psi_1' P_1 t - \frac{R_{x_1}}{m_k} t,\\ \dot{z}_1 &= \psi_2' P_1 t - \frac{G_k}{m_k} t - \frac{R_{z_1}}{m_k} t,\end{aligned}\right\} \quad (71)$$

$$\omega t \in \left[\left(2k - 1 \right) \pi, \quad 2k\pi \right], \; k = 1, 2, ...$$

By substituting the values of the arbitrary constants C_1, L_1, C_2 and L_2 into the system of equations (69), we obtain:

$$\left.\begin{aligned}x_1 &= \psi_1' P_1 \frac{t^2}{2} - \frac{R_{x_1} t^2}{2m_k} + x_{10},\\ z_1 &= \psi_2' P_1 \frac{t^2}{2} - \frac{G_k t^2}{2m_k} - \frac{R_{z_1} t^2}{2m_k} - \frac{1}{3} h_k,\end{aligned}\right\} \quad (72)$$

$$\omega t \in \left[\left(2k - 1 \right) \pi, \quad 2k\pi, \right], \; k = 1, 2, ...$$

The systems of equations (71) and (72), respectively, characterize the laws of variation of the speed and displacement of the beet root's centre of mass in the process of its direct lifting from the soil in the absence of the perturbing force action.

Now we are going to derive the differential equation of the beet root's rotation around its centre of mass (around the axis Cy, which passes through the beet root's centre of mass (point C) parallel to the axis O_1y_1). According to (Dreizler & Lüdde, 2010), the said equation will have the following form:

$$I_{yc} \frac{d^2\theta}{dt^2} = M_{yc}^e, \quad (73)$$

where θ is the angular displacement of the beet root around the axis Cy_c; I_{yc} is the root's moment of inertia with reference to the axis Cy_c; M_{yc}^e is the moment of rotation around the axis Cy_c (total moment of all external forces applied to the beet root with reference to the axis Cy_c).

Further, let's find the moments of all external forces with reference to the axis Cy_c in accordance with the schematic model of forces presented in Fig. 1. As the movement of the beet root's centre of mass is considered with reference to the coordinate system $x_1O_1y_1z_1$, so we will determine the positions of K_1 and K_2 – the points of contact between the root and the lifting shares' working surfaces $A_1B_1C_1$ and $A_2B_2C_2$ with reference to the same coordinate system. As we can see in the schematic model in Fig. 1, the ordinate of the contact points K_1 and K_2 in the assumed coordinate system will be equal to:

$$z_{K1} = z_{K2} = -h_k + h,$$

where h is the distance from the conditional fixation point O to the plane that extends through the contact points and is perpendicular to the beet root symmetry axis.

Since the movement of the vibrational lifting unit shares takes place at a certain depth, the value h for the specific beet root can vary only within the share oscillation amplitude, which is considerably smaller in comparison with the value h. Therefore, the value h for any specific beet root can be regarded constant. The ordinate of the beet root's centre of mass (point C) at a random instant will be:

$$z_c = z_1,$$

where z_1 is determined by the second equation of the system (64).

Thus, the ordinate of the points K_1 and K_2 varies from the ordinate of the point C by the value:

$$-h_k + h - z_1.$$

So, for example, from the very beginning of the direct lifting $\left(z_1 = -\frac{h_k}{3}\right)$ we have:

$$-h_k + h + \frac{h_k}{3} = h - \frac{2h_k}{3}.$$

Then the moments of all external forces applied to the beet root at a random instant will be equal to:

$$M_{yc}(\bar{Q}_{p1}) = M_{yc}(\bar{Q}_{p2}) = -Q_{p1}(-h_k + h - z_1)\sin\theta \tag{74}$$

since the force vectors $\overline{Q}_{p1}$ and $\overline{Q}_{p2}$ are parallel to the plane $x_1O_1z_1$.

$$M_{yc}(\bar{P}_1) = M_{yc}(\bar{P}_2) = -P_1(-h_k + h - z_1)\cos\theta, \tag{75}$$

since the force vectors $\overline{P}_1$ and $\overline{P}_2$ are parallel to the plane $x_1O_1z_1$.

$$M_{yc}(\bar{F}_1) = M_{yc}(\bar{F}_2) = F_1\cos\gamma_k\,(-h_k + h - z_1)\sin\theta\,, \tag{76}$$

or, taking into account the expression (26), we have:

$$\begin{aligned} M_{yc}(\bar{F}_1) = M_{yc}(\bar{F}_2) \\ &= \left(\frac{1}{2}fH\cos\delta\sin(\omega t) \right. \\ &\quad \left. + fP_1\sin\gamma\right)\sin[\gamma + \alpha_{K1max}\sin(\omega t)]\cos\gamma_k\,(-h_k + h \\ &\quad - z_1)\sin\theta\,, \\ \omega t \in [2k\pi, (2k+1)\pi], \qquad k = 0, 1, 2, \ldots \end{aligned} \tag{77}$$

Then, taking into consideration the expression (29), we obtain:

$$\begin{gathered} M_{yc}(\bar{F}_1) = M_{yc}(\bar{F}_2) = fP_1\sin^2\gamma\cos\gamma_k\,(-h_k + h - z_1)\sin\theta\,, \\ \omega t \in [(2k-1)\pi, 2k\pi], \qquad k = 1, 2, \ldots \end{gathered} \tag{78}$$

$$M_{yc}(\bar{E}_1) = M_{yc}(\bar{E}_2) = E_1\cos\gamma\,(-h_k + h - z_1)\cos\theta\,. \tag{79}$$

Considering the expression (27), we obtain:

$$\begin{aligned} M_{yc}(\bar{E}_1) = M_{yc}(\bar{E}_2) \\ &= \left(\frac{1}{2} fH\cos\delta\sin(\omega t) + f\,P_1\sin\gamma\right) \\ &\times\cos[\gamma+\alpha_{K1max}\sin(\omega t)]\cos\gamma_k(-h_k+h-z_1)\cos\theta\,, \\ &\omega t\in[2k\pi,(2k+1)\pi], \qquad k=0,1,2,\ldots \end{aligned} \tag{80}$$

And after using the expression (30), we will come to:

$$\begin{gathered} M_{yc}(\bar{E}_1) = M_{yc}(\bar{E}_2) = \frac{1}{2} fP_1\sin 2\gamma\cos\gamma\,(-h_k+h-z_1)\cos\theta\,, \\ \omega t\in[(2k-1)\pi,2k\pi,], \qquad k=1,2,\ldots \end{gathered} \tag{81}$$

For the remaining forces:

$$M_{yc}(\bar{G}_k) = 0\,, \tag{82}$$

$$M_{yc}(\bar{R}_{x1}) = 0\,, \tag{83}$$

$$M_{yc}(\bar{R}_{z1}) = 0\,, \tag{84}$$

since the vectors $\overline{\boldsymbol{G}}_k$, $\overline{\boldsymbol{R}}_{x1}$ and $\overline{R}_{z1}$ intersect the axis Cy$_c$.

Hence, basing on the expressions (74), (75), (77) or (78), (80) or (81), (82), (83), (84) and the moment *M* of the couple of forces of the loosened soil's resistance to the rotation of the beet root, we find the value of the rotation moment $M^e_{y_c}$ of all external forces with reference to the axis Cy as follows:

$$\begin{aligned} M^e_{yc} = -2Q_{p1}(-h_k+h-z_1)\sin\theta + 2P_1\cos\theta\,(-h_k+h-z_1) \\ &+ (fH\cos\delta\sin(\omega t) + 2f\,P_1\sin\gamma) \\ &\times\sin[\gamma+\alpha_{K1max}\,sin(\omega t)] \\ &\times\cos[\gamma+\alpha_{K1max}\sin(\omega t)]\cos\gamma_k(-h_k+h-z_1)\cos\theta\,, \end{aligned} \tag{85}$$

or, after some transformations:

$$\begin{aligned} M^e_{yc} = 2P_1\cos\theta\,(-h_k+h-z_1) \\ &+ 2fP_1\sin^2\gamma\cos\gamma_k\,(-h_k+h-z_1)\sin\theta \\ &+ fP_1\sin 2\gamma\cos\gamma\,(-h_k+h-z_1)\cos\theta - M\,, \\ &\omega t\in[(2k-1)\pi,2k\pi], \qquad k=1,2,\ldots \end{aligned} \tag{86}$$

The moment of inertia I_{yc} of the beet root with reference to the axis Cy$_c$ is determined with the use of the expression stated in (Bulgakov, 2011):

$$I_{yc} = \left(0.038 + 0.15\tan^2\gamma_k\right). \tag{87}$$

By substituting the expressions (2), (87), (85) or (86) into the differential equation (73) we obtain the differential equation of the beet root's rotation around the axis Cy$_c$ during its direct lifting from the soil, which has the following form:

$$\begin{aligned}(0.038 + 0.25\tan^2\gamma_k)m_k h_k^2 \frac{d^2\theta}{dt^2} \\ &= -H(-h_k + h - z_1)\sin\theta\sin(\omega t) \\ &+ 2P_1\cos\theta\,(-h_k + h - z_1) \\ &+ (fH\cos\delta\sin(\omega t) \\ &+ 2fP_1\sin\gamma)\sin[\gamma + \alpha_{K1max}\sin(\omega t)]\cos\gamma_k\,(-h_k \\ &+ h - z_1)\sin\theta + (fH\cos\delta\sin(\omega t) \\ &+ 2fP_1\sin\gamma)\cos[\gamma{+}\alpha_{K1max}\sin(\omega t)]\cos\gamma\,(-h_k + h \\ &- z_1)\cos\theta - M\,, \\ &\omega t \in [2k\pi, (2k+1)\pi], \qquad k = 0, 1, 2, \ldots\end{aligned} \tag{88}$$

or:

$$\begin{aligned}(0.038 + 0.15\tan^2\gamma_k)m_k h_k^2 \frac{d^2\theta}{dt^2} \\ &= 2P_1\cos\theta\,(-h_k + h - z_1) + 2fP_1\sin^2\gamma\cos\gamma_k\,(-\,h_k \\ &+ h \\ &- z_1)\sin\theta \\ &+ fP_1\sin(2\gamma)\cos\gamma\,(-h_k + h - z_1)\cos\theta - M\,, \\ &\omega t \in [(2k-1)\pi, 2k\pi], \qquad k = 1, 2, \ldots\end{aligned} \tag{89}$$

The initial conditions for the obtained differential equation (89) are established basing on the same considerations as for the differential equation (52) and they will have the following form:

At $t = 0$:

$$\theta = 0, \quad \dot{\theta} = 0\,. \tag{90}$$

The differential equation (88) is nonlinear. It can be solved only with the use of numerical techniques and a PC. With this approach, the value z_1 for each cycle of using the numerical algorithm has to be obtained from the second equation of the system (64) for the respective instant t_k.

The differential equation (89) is also nonlinear, since it includes the value z_1, which is a variable, and for any instant t_k this value z_1 has to be obtained from the second equation of the system (72).

Thus, the obtained analytic expressions, essentially, constitute the theory of direct sugar beet root lifting from the soil with the use of vibrational lifting units. The reached analytic expressions make it possible to define the kinematic modes of vibration-assisted beet root lifting basing on the requirement of keeping the roots intact and the design parameters of the vibrational lifting unit.

RESULTS AND DISCUSSION

Now, let's apply the achieved results of the developed theory and construct an algorithm for computing the kinematic parameters of the work process under consideration. Here are its main provisions:

1. First we specify the required initial data for the calculation.
2. Then we find the values φ_1, ψ_1, φ_2, ψ_2 in accordance with the expressions (55), (56), (57) and (58), respectively.
3. Next, we find the sugar beet root motion law during its direct lifting from the soil, according to the expression (64).
4. Now we move to drawing the diagrams for various values of the initial parameters, from those diagrams we find the time of duration of the direct beet root lifting from the soil.
5. In order to carry out the numerical calculations, we have to specify the required parameters. Thus, according to Pogorely & Tatyanko, (2004) and Bulgakov, (2011), the specified parameters have the following values:

- (average) mass of a sugar beet root: $m_k = 0.9$ kg;
- (average) length of a sugar beet root: $h_k = 0.25$ m;
- angles of the vibrational lifting unit's trihedral wedges: $\gamma = 14°$, $\beta = 52°$;
- friction coefficient of steel on the sugar beet root surface: $f = 0.45$;
- resistance force exerted by the soil, when a sugar beet root moves in it: $R_x = 100$ N, $R_z = 100$ N;
- amplitude of perturbing force: $H = 500$ N;
- transverse moving force: $P_1 = 400$ N;
- angle of deflection of the friction force vector from the vector of its minimal value: $\alpha_{K1\max} = 30°$;
- initial position of the sugar beet root's centre of mass on the axis O_1x_1: $x_{10} = 0.2$ m.

The dihedral angle δ between the wedge's working surface and the lower base of the lifting share can be derived from the expression stated in (Bulgakov, 2011): $\delta = arctg\dfrac{\cos\beta}{\sin\delta\cos\gamma}$.

Calculations have been carried out for several values of the vibrational lifting unit oscillation frequency.

Basing on the obtained law of motion of the beet root's centre of mass (64) in the system of coordinates $x_1O_1z_1$, we draw the graphs $x_1 = x_1(t)$, $z_1 = z_1(t)$ in the MathCAD environment (Fig. 2) in order to determine the lifting time.

As may be inferred from the graphs, the duration of the beet root lifting from the soil ($z_1 = 0$) reaches only 0.032 s.

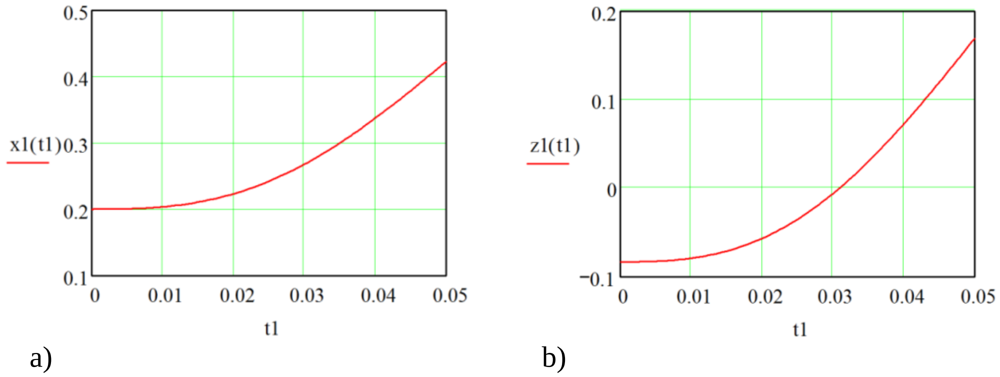


Figure 2. Graphs of the root's centre of mass displacement along the axes O_1x_1 (a) and O_1z_1 (b) as a function of time during the direct beet root lifting from the soil (H = 500 N; P_1 = 400 N; R_x = 100 N; R_z = 100 N; v = 10 Hz).

In Fig. 3, the motion trajectory of the beet root's centre of mass during the direct beet root lifting from the soil is shown.

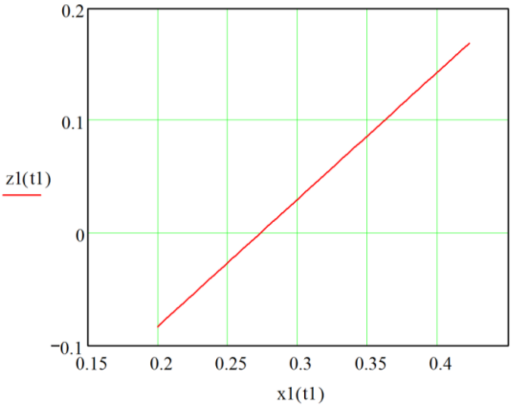


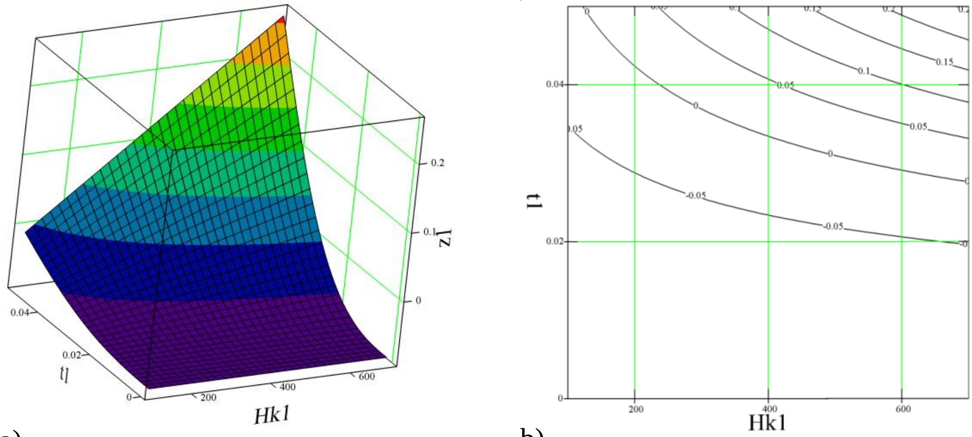
Figure 3. Beet root motion trajectory in the coordinate system $x_1O_1z_1$ during the direct lifting of the root from the soil: (H = 500 N, P_1 = 400 N, R_x = 100 N, R_z = 100 N, v = 10 Hz).

It becomes evident from the presented graph that within the interval of lifting the beet root from the soil ($-0.083 \leq z_1 \leq 0$) its centre of mass moves effectively on a straight line.

Obviously, this motion trajectory represents the actual trajectory of motion of the beet root's centre of mass only as a certain approximation, since the soil resistance forces during the beet root displacement R_{x1} and R_{z1} are assumed to have constant magnitudes.

Also, calculations have been carried out for the displacement of the beet root's centre of mass along the axis O_1z_1 until its complete lifting from the soil as a function of the changing perturbing force amplitude and $z_1 = z_1(H,t)$ at P_1 = const and $z_1 = z_1(P,t)$ at P = const have been obtained.

In Fig. 4, the surface and profile graph of $z_1 = z_1(H,t)$ subject to the perturbing force amplitude variation within a range of H = 100…700 N (for a transverse moving force value of P_1 = 400 N and an oscillation frequency value of v = 10 Hz) are presented.

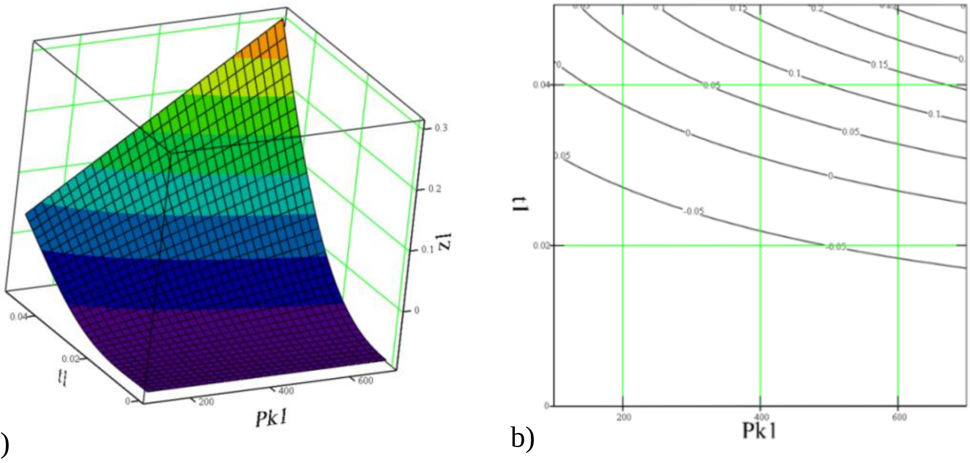


a) b)

Figure 4. Surface (a) and profile graph (b) of function $z_1 = z_1(H,t)$ for the perturbing force amplitude's variation within a range of H = 100…700 N (P_1 = 400 N, v = 10 Hz).

As one may see in the shown graph, when the perturbing force amplitude changes within a range of 100…700 N, the time of beet root lifting from the soil changes within an interval of 0.053…0.028 s.

In Fig. 5, the surface and profile graph of the function $z_1 = z_1(P_1,t)$ subject to the transverse moving force variation within a range of P_1 = 100…700 N (for a perturbing force amplitude value of H = 500 N and an oscillation frequency value of v = 10 Hz) are presented.



a) b)

Figure 5. Surface (a) and profile graph (b) of function $z_1 = z_1(P_1,t)$ for the transverse moving force variation within a range of P_1 = 100…700 N (H = 500 N, v = 10 Hz).

As may be inferred from the shown graph, when the transverse moving force changes within a range of 100…700 N, the time of beet root lifting from the soil changes within a range of 0.043…0.026 s.

CONCLUSIONS

A new theory of lifting sugar beet roots from the soil with vibrational lifting units has been worked out. This includes the analytic description of the work process at all stages of lifting, starting from the instant, when the vibrational lifting unit grips the root, and up to the complete lifting of the root out of soil.

The system of differential equations for the motions of the root during its direct lifting from the soil has been obtained. Solving the obtained differential equation system allows to find the law of motion of the sugar beet root's centre of mass in the analytical form.

The calculations in the MathCAD environment performed on a PC have made it possible to find the duration of direct beet root lifting from the soil and analyse the effect that the vibrational lifting unit design parameters and the kinematic modes of performing the work process have on the lifting time.

Thus, at a perturbing force amplitude of $H = 500$ N, a transverse moving force of $P_1 = 400$ N, soil resistance forces: along the axis Ox_1 – $R_x = 100$ N and along the axis Oz_1 – 100 N, a perturbing force frequency of $\nu = 10$ Hz the time of root lifting from the soil is 0.032 s. When the perturbing force amplitude varies within a range of 100…700 N (at a transverse moving force of $P_1 = 400$ N and a perturbing force frequency of $\nu = 10$ Hz), the time of beet root lifting from the soil varies within a range of 0.053…0.028 s.

When the transverse moving force varies within a range of $P_1 = 100…700$ N (at a perturbing force amplitude of $H = 500$ N and a perturbing force frequency of 10 Hz) the time of beet root lifting from the soil varies within a range of 0.043…0.026 s.

The achieved results of the theoretical research provide a possibility to determine the optimal kinematic modes of operation and vibrational lifting unit design parameters, proceeding from the requirement of keeping sugar beet roots intact when harvesting them.

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