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Natural vibrations of stepped arches with cracks

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Abstract. Natural vibrations of elastic circular arches are studied. The arches are assumed to be of constant width and piece wise constant height. It is assumed that at the re-entrant corners of steps stable surface cracks are located. The aim of the paper is to assess the sensitivity of the eigenfrequencies on the geometrical and physical parameters of the arch including the length and location of each crack.

Key words: elasticity, arch, natural vibrations, crack, eigenfrequency.

INTRODUCTION

The problems of vibration and stability of beams, plates and shells have a great importance in the civil and engineering. Vibration of curved beams is studied by several researches (see Vinson & Sierakowski, 2002; Qatu, 2004; Reddy, 2004). The natural and forced vibrations of beams weakened with the crack-like defects have been investigated by Rizos at al. (1990), Dimarogonas (1996), Nandwana & Maiti (1997), Chondros et al. (1998), Kisa & Brandon (2000) and others. Lellep & Kägo (2011; 2013) investigated the influence of cracks on eigenfrequencies of elastic stretched strips and plates.

In the previous papers by Lellep & Liyvapuu (2015a; 2015b) vibrations of elastic arches made of homogeneous and laminated materials were studied.

Due to the practical needs the investigations of the free and forced vibrations of beams, arches, plates and shells are carried out by many investigators (see Qatu 2004; Soedel 2004). During last years new approach to the free vibration analysis are developed in the papers by Eroglu (2015), Wu & Chiang (2004) for the case of in-plane vibrations. While Ishaguddin et al. (2016) and Kawakami et al. (1995) accounted for the out-of-plane vibrations in their studies, Sadeghpour et al. (2016) considered the effect of debonding during the process of natural vibrations.

In the paper by Wu & Chiang (2004) the effect of both, the shear deformation and rotatory inertia are included in the analysis using finite arch elements.

Although usually the in-plane and out-of-plane vibrations of beams and bars are tackled separately the approach by Wu and Chiang admits to consider the both versions from the common point of view.

In the present paper we are interested in the evaluation of the influence of cracks on the natural frequencies of arches. That is why the simplest theory of vibration of beams is employed.

Here results of the previous study Lellep & Liyvapuu (2015b) are extended to the case of a stepped arch weakened with non-penetrated surface cracks. The cracks are assumed to be stable surface cracks. The problems of propagation of cracks are outside the scope of the present paper.

MATERIALS AND METHODS

Problem formulation

Let us study the free vibrations of a circular arch of radius R. It is assumed that the arch has rectangular cross section with dimensions b (the width) and the total height H. The total height is assumed to be piece wise constant, e.g.

$$h = h_j, \qquad \varphi \in (\alpha_j, \alpha_{j+1}) \tag{1}$$

for $j = 0, \dots, n$.

In (1) φ stands for the current angle (Fig. 1.) and $h_0, \dots, h_n$ and $\alpha_0, \dots, \alpha_n$ are given constants.

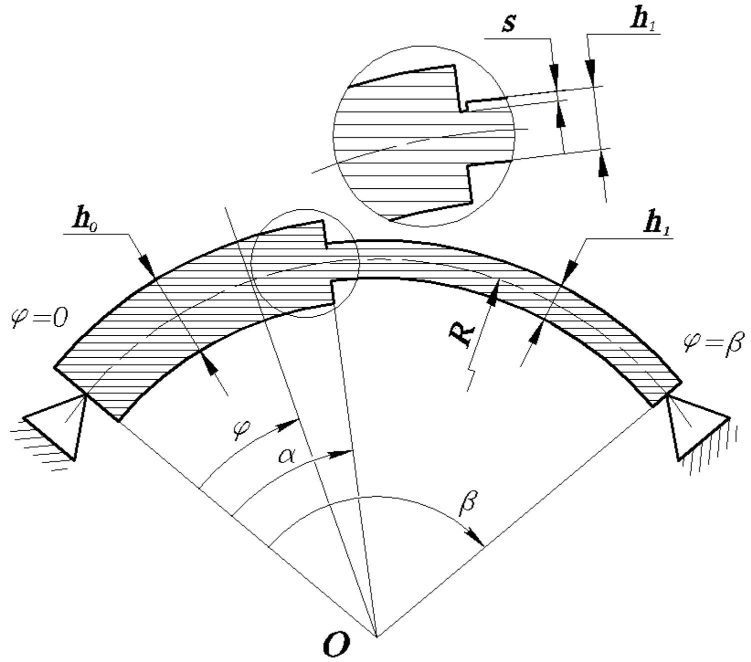

Figure 1. Simply supported stepped arch with a crack.

Here $\alpha_0 = 0$ and $\alpha_{n+1} = \beta$.

The arch is simply supported at $\varphi = 0$ and $\varphi = \beta$.

The arch is weakened with cracks located at the re-entrant corners of steps. It is assumed that the crack located at the position $\varphi = \alpha_j$ has the length c_j. Evidently, the eigenfrequencies of the arch depend on the geometry of the arch and on the geometry of the crack.

The aim of the paper is to determine the eigenfrequencies of the arch and to study the sensitivity of the eigenfrequencies on the geometrical and physical parameters of the arch.

Basic equations and assumptions

Treating the equilibrium of an element of the vibrating arch one can conclude that (see Soedel 2004; Lellep & Liyvapuu 2015a; Lellep & Liyvapuu 2015b).

$$M'' + M - \bar{\rho} h_j R^2 \ddot{W} = 0 \tag{2}$$

for $\varphi \in (\alpha_j,\ \alpha_{j+1})$ $j = 0, \ldots, n$. Here M stands for the bending moment, W is the transverse displacement (deflection) and $\bar{\rho}$ is the material density. In the case of a composite or laminated material the quantity $\bar{\rho}$ is the average of densities of the layers (see Reddy 2004; Qatu 2004).

Here and henceforth

$$M' \equiv \frac{\partial M}{\partial \varphi}, \quad \dot{W} \equiv \frac{\partial W}{\partial t}, \tag{3}$$

t standing for time.

According to the Hook's law one has (see Lellep & Liyvapuu 2015a; Lellep & Liyvapuu 2015b)

$$M = D_j \varkappa \tag{4}$$

for $\varphi \in (\alpha_j,\ \alpha_{j+1})$ $j = 0, \ldots, n$. Here

$$\varkappa = -\frac{1}{R^2}(W + W''). \tag{5}$$

Because we are interested in evaluation of the influence of cracks on the natural frequencies we need the simplest theory of vibration. That is why it is assumed herein that the axial extension $\varepsilon = 0$ and therefore, $U' = -W$.

Here U stands for the axial displacement. Note that in the case of any homogeneous material

$$D_j = \frac{E h_j^3}{12(1 - \nu^2)}, \tag{6}$$

where E is the Young modulus and ν – the Poisson ratio.

Assuming that both ends of the arch are simply supported one can present the boundary conditions as

$$W(0, t) = 0; \quad M(0, t) = 0 \tag{7}$$

and

$$W(\beta,t) = 0; \qquad M(\beta,t) = 0\,. \tag{8}$$

Substituting (4) and (5) in the equilibrium equation (2) leads to the equation

$$\frac{D_j}{R^2}\left(W^{IV} + 2W'' + W\right) + \bar{\rho} h_j R^2 \ddot{W} = 0 \tag{9}$$

for $\varphi \in \left(\alpha_j,\ \alpha_{j+1}\right)\ j = 0, \dots, n$.

The arch under consideration has stable surface cracks at $\varphi = \alpha_j$. It is well known that defects deteriorate the mechanical behaviour of structures. The influence of cracks on the natural vibrations of arches is modelled by the method suggested by Chondros at al. (1998) and Dimarogonas (1998). According to this method the slope of the deflection is considered as a discontinuous quantity at the cross sections with cracks. Let us denote

$$\theta_j = W'\left(\alpha_j + 0, t\right) - W'(\alpha_j - 0, t)\,. \tag{10}$$

It was shown in Lellep & Kägo (2013) and Lellep & Liyvapuu (2015b) that on can take

$$\theta_j = p_j \varkappa\left(\alpha_j + 0, t\right), \tag{11}$$

where

$$p_j = \frac{6\pi h_j}{1-\nu^2}\, f(s_j) \tag{12}$$

and

$$\begin{aligned} f\left(s_j\right) = 1.86 s_j^2 - 3.95 s_j^3 + 16.37 s_j^4 - 34.23 s_j^5 + 76.81 s_j^6 - \\ -126.93 s_j^7 + 172 s_j^8 - 143.97 s_j^9 + 66.56 s_j^{10}\,. \end{aligned} \tag{13}$$

Solution of governing equations

The equation (9) is a linear fourth order equation with partial derivatives. Making use of the method of separation of variables (see Soedel 2004; Lellep & Liyvapuu 2015a; Lellep & Liyvapuu 2015b) one can look for the solution of (9) in the form

$$W(\varphi,t) = w(\varphi)\cdot\sin(\omega t). \tag{14}$$

In (14) the first term in the right hand side of the equality is assumed to be a function of the variable φ. Substituting (14) in (9) leads to the ordinary differential equation of the fourth order

$$\frac{D_j}{R^2}\left(w^{IV} + 2w'' + w\right) + \bar{\rho} h_j R^2 \omega^2 w = 0 \tag{15}$$

for $\varphi \in \left(\alpha_j,\ \alpha_{j+1}\right)\ j = 0, \dots, n$.

Evidently, the general solution of (15) can be presented as

$$w = C_{1j}\cosh(\mu_j\varphi) + C_{2j}\sinh(\mu_j\varphi) + C_{3j}\cos(\nu_j\varphi) + C_{4j}\sin(\nu_j\varphi) \quad (16)$$

where C_{1j} — C_{4j} are arbitrary constants and

$$\mu_j = \sqrt{1 - \omega R^2\sqrt{\frac{\bar{\rho}h_j}{D_j}}}, \qquad \nu_j = \sqrt{1 + \omega R^2\sqrt{\frac{\bar{\rho}h_j}{D_j}}}. \quad (17)$$

According to (7), (8) and (14) one can present the boundary conditions for $w(\varphi)$ as

$$w(0) = 0, \qquad w''(\beta) = 0 \quad (18)$$

and

$$w(\beta) = 0, \qquad w''(\beta) = 0. \quad (19)$$

The boundary conditions (18) with (16) furnish the relations

$$\begin{aligned} &C_{10} + C_{30} = 0, \\ &\mu_0^2 C_{10} - \nu_0^2 C_{30} = 0. \end{aligned} \quad (20)$$

It immediately follows from (20) that

$$C_{10} = C_{30} = 0, \quad (21)$$

provided $\mu_0^2 + \nu_0^2 \neq 0$.

The boundary requirements (19) lead to the equations

$$\begin{aligned} &C_{1n}\cosh(\mu_n\beta) + C_{2n}\sinh(\mu_n\beta) + C_{3n}\cos(\nu_n\beta) + C_{4n}\sin(\nu_n\beta) = 0, \\ &\mu_n^2(C_{1n}\cosh(\mu_n\beta) + C_{2n}\sinh(\mu_n\beta)) - \nu_n^2(C_{3n}\cos(\nu_n\beta) + \\ &+C_{4n}\sin(\nu_n\beta)) = 0, \end{aligned} \quad (22)$$

provided $\mu_n^2 + \nu_n^2 \neq 0$.

The particular solution of (15) must be constructed so that in each segment the solution is given by (16) and at the boundary the requirements (18), (19) are taken into account.

Moreover, at $\varphi = \alpha_j$ the quantities W, M and $Q = M'$ must be continuous; the slope W' must satisfy (10) — (13). Thus, $W,\ D(W'' + W)$ and $D(W''' + W')$ are continuous. Here

$$D(\alpha_j - 0) = D_j; \qquad D(\alpha_j + 0) = D_{j+1} \tag{23}$$

for each $j = 1, \dots, n$.

The continuity conditions can be presented as

$$\begin{aligned} C_{1j}\cosh(\mu_j\alpha_j) + C_{2j}\sinh(\mu_j\alpha_j) + C_{3j}\cos(\nu_j\alpha_j) + C_{4j}\sin(\nu_j\alpha_j) \\ = C_{1j+1}\cosh(\mu_{j+1}\alpha_j) + C_{2j+1}\sinh(\mu_{j+1}\alpha_j) \\ + C_{3j+1}\cos(\nu_{j+1}\alpha_j) + C_{4j+1}\sin(\nu_{j+1}\alpha_j)\ ; \end{aligned}$$

$$\begin{aligned} \mu_{j+1}\big(C_{1j+1}\sinh(\mu_{j+1}\alpha_j) + C_{2j+1}\cosh(\mu_{j+1}\alpha_j)\big) \\ + \nu_{j+1}\big(-C_{3j+1}\sin(\nu_{j+1}\alpha_j) + C_{4j+1}\cos(\nu_{j+1}\alpha_j)\big) = \\ = \mu_j\big(C_{1j}\sinh(\mu_j\alpha_j) + C_{2j}\cosh(\mu_j\alpha_j)\big) \\ + \nu_j\big(-C_{3j}\sin(\nu_j\alpha_j) + C_{4j}\cos(\nu_j\alpha_j)\big) \\ - \frac{p_j D_j}{R^2}\left\{C_{1j}\big(1+\mu_j^2\big)\cosh(\mu_j\alpha_j) + C_{2j}\big(1+\mu_j^2\big)\sinh(\mu_j\alpha_j)\right. \\ \left. + C_{3j}\big(1-\nu_j^2\big)\cos(\nu_j\alpha_j) + C_{4j}\big(1-\nu_j^2\big)\sin(\nu_j\alpha_j)\right\}; \end{aligned}$$

$$\begin{aligned} D_j\left\{C_{1j}\big(1+\mu_j^2\big)\cosh(\mu_j\alpha_j) + C_{2j}\big(1+\mu_j^2\big)\sinh(\mu_j\alpha_j)\right. \\ \left. + C_{3j}\big(1-\nu_j^2\big)\cos(\nu_j\alpha_j) + C_{4j}\big(1-\nu_j^2\big)\sin(\nu_j\alpha_j)\right\} \\ = D_{j+1}\left\{C_{1j+1}\big(1+\mu_{j+1}^2\big)\cosh(\mu_{j+1}\alpha_j)\right. \\ + C_{2j+1}\big(1+\mu_{j+1}^2\big)\sinh(\mu_{j+1}\alpha_j) \\ + C_{3j+1}\big(1-\nu_{j+1}^2\big)\cos(\nu_{j+1}\alpha_j) \\ \left. + C_{4j+1}\big(1-\nu_{j+1}^2\big)\sin(\nu_{j+1}\alpha_j)\right\}; \end{aligned} \tag{24}$$

$$\begin{aligned} D_j\left\{C_{1j}\big(\mu_j+\mu_j^3\big)\cosh(\mu_j\alpha_j) + C_{2j}\big(\mu_j+\mu_j^3\big)\sinh(\mu_j\alpha_j)\right. \\ \left. + C_{3j}\big(-\nu_j-\nu_j^3\big)\sin(\nu_j\alpha_j) + C_{4j}\big(\nu_j-\nu_j^2\big)\cos((\nu_j\alpha_j)\right\} \\ = D_{j+1}\left\{C_{1j+1}\big(\mu_{j+1}+\mu_{j+1}^3\big)\cosh(\mu_{j+1}\alpha_j)\right. \\ + C_{2j+1}\big(\mu_{j+1}+\mu_{j+1}^3\big)\sinh(\mu_{j+1}\alpha_j) \\ + C_{3j+1}\big(-\nu_{j+1}-\nu_{j+1}^3\big)\sin(\nu_{j+1}\alpha_j) \\ \left. + C_{4j+1}\big(\nu_{j+1}-\nu_{j+1}^3\big)\cos(\nu_{j+1}\alpha_j)\right\}. \end{aligned}$$

NUMERICAL RESULTS AND DISCUSSION

The system of equations (24) augmented with (21) and (23) present a system of equation for determination of unknown C_{ij} where $i = 1, \dots, 4$ and $j = 0, \dots, n$. This system consists of $4n + 4$ equations with the same number of unknowns. Since the system is a linear homogeneous system a non-trivial solution exists if it determinant Δ vanishes.

The solution of equation $\Delta = 0$ admits to define the eigenfrequencies. The solution procedure is implemented with the aid of the computer code MATLAB. The results of calculations are presented in Figs 2–5 for the arch with a single step ($n = 1$) and $R = 1m,\ h_0 = 0.02m,\ h_1 = 0.01m$.

The material of the arch is a mild steel with $E = 2.1 \cdot 10^{11} Pa,\ \nu = 0.3$.

The natural frequencies of an arch without any defects compare favorably with those obtained by the finite element method in the case of higher modes (see Zheng & Fan, 2003). For instance, according to the previous study $\omega_3 = 2{,}160,\ \omega_5 = 6{,}480$, whereasthe predictions obtained by the finite element method are $\overline{\omega}_3 = 3{,}933,\ \overline{\omega}_5 = 8{,}150$. The discrepancies between these predictions are caused by the simplified model of the present problem. Due to the hypotheses made above the comparison has no sense for the first mode. Evidently the method leads to crude approximations in the case of lower modes of deformation and deeper arches.

The influence of the first natural frequency on the location of the step is illustrated in Fig. 2 for the elastic arch with $\beta = 1$. Different curves in Fig. 2 correspond to different values of the crack depth. The upmost curve in Fig. 2 correspond s to the arch without any defects. It can be seen from Fig. 2 that the highest values of the natural frequency are obtained in the case of arch which is free of cracks.

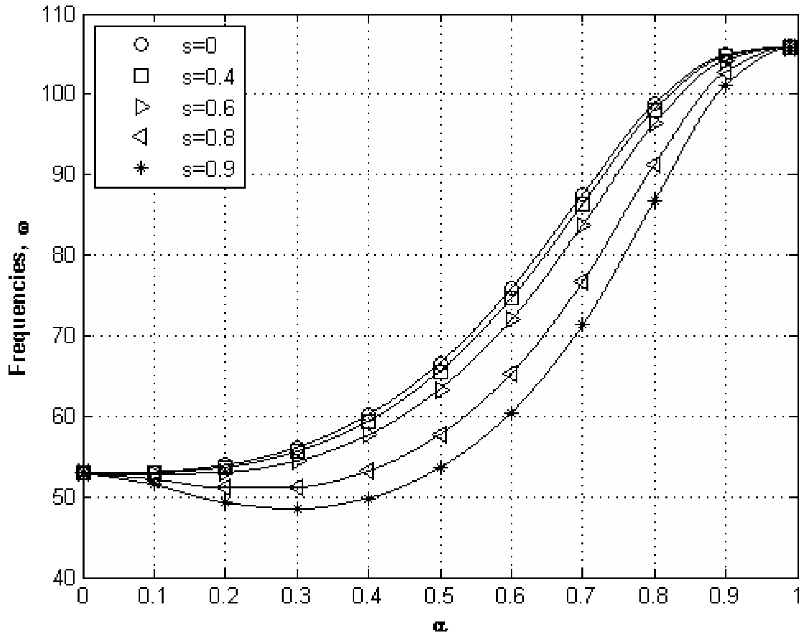

Figure 2. Natural frequency of the arch vs. depth of the crack.

The natural frequency versus the step location is depicted in Figs 3–5 for different values of the crack length. Different curves in Figs 3–5 correspond to the arches with the central angle $\beta = 1.0;\ \beta = 1.2;\ \beta = 1.5;\ \beta = 1.6;\ \beta = 1.7$ and $\beta = 1.8$, respectively. Note that Fig. 3 is assotiated with the arch which has no any defect. It can be seen from Fig. 3 that the larger is the central angle of the arch, the lower is the natural frequency as might be expected. Note that similar relationship between the length and the natural frequency takes place in the case of straight beams, as well. In the case of beams it reads: the longer is the beam the lower is the natural frequency. Similar results are presented in Fig. 4 and Fig. 5 for arches with crack lengthes $c = 0.6h_1$ and $c = 0.8h_1$, respectively.

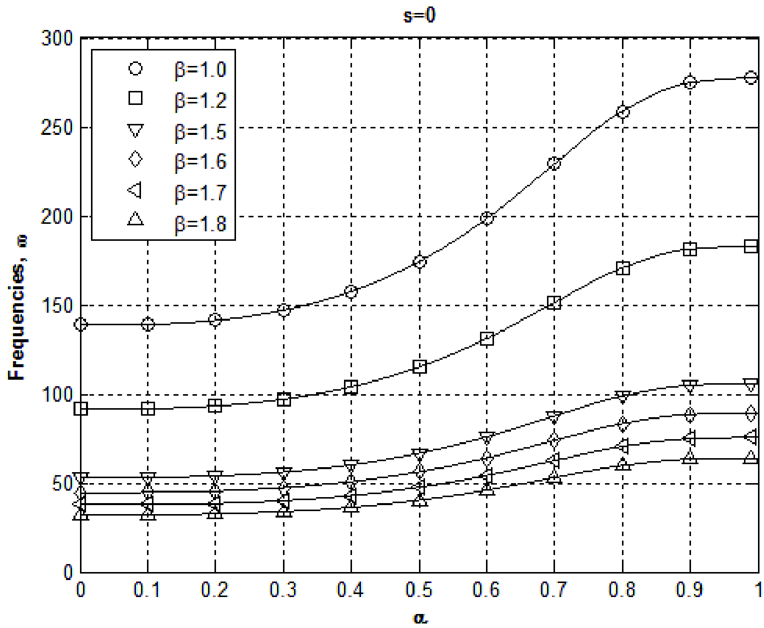

Figure 3. Natural frequency versus step location (s = 0).

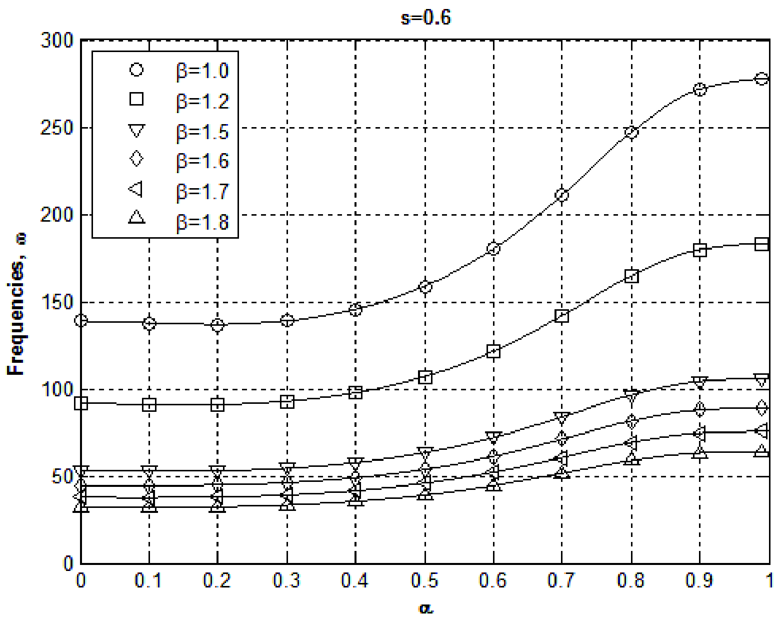

Figure 4. Natural frequency versus step location (s = 0.6).

It can be seen from Fig. 5 that the upper curves associated with $\beta = 1.0$ and $\beta = 1.2$ are decreasing in the range of small values of α. If, however, $\alpha > 0.4$ the function $\omega = \omega(\alpha)$ are increasing everywere. In the particular case if $h_0 = h_1$ the natural frequency ω decreases monotonically with increasing value of β (see Lellep & Liyvapuu 2015a; Lellep & Liyvapuu 2015b).

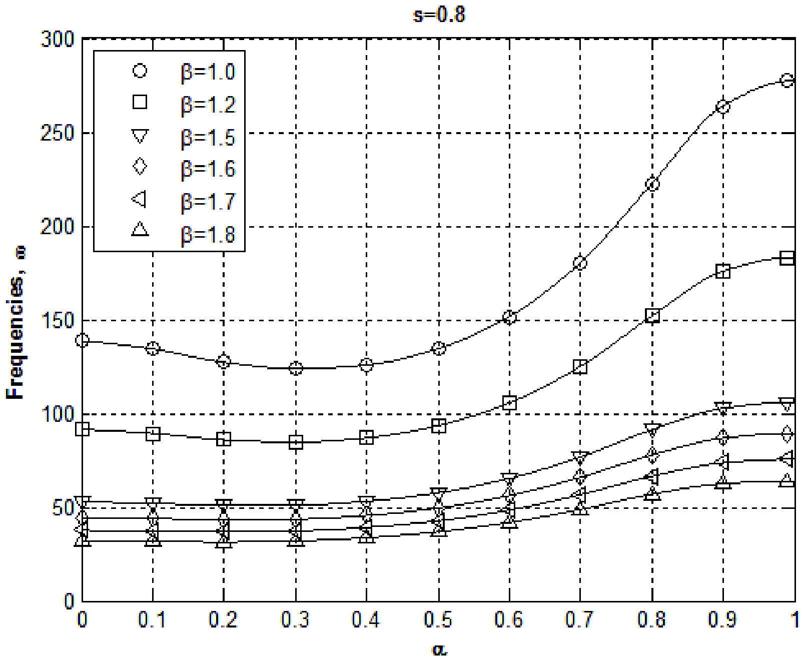

Figure 5. Natural frequency versus step location (s = 0.8).

CONCLUSIONS

Natural vibrations of circular arches with piece wise constant thickness have been considered. An analytical method for determination of eigenfrequencies of arches with cracks was developed. Comparison of the results of the present study with the numerical predictions shows that the present model leads to more accurate predictions in the case of higher deformation modes. It was shown that the parameters of the crack essentially influence on the vibration of the arch. The highest value of the natural frequency corresponds to the arch with any defects.

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