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Theoretical study on forced transverse oscillations of root in soil with provision for soil's elastic and damping properties

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Abstract. The topic of the paper is the theory of the forced transverse oscillations performed by the root fixed in the soil under the action of the harmonic perturbing force vectored at right angle to the root's centreline and along the line of the translational motion performed by the lifter. On the basis of applying the Ostrogradsky-Hamilton variational principle and using the equivalent schematic model developed by the authors, the expressions have been obtained that allow to determine the amplitude of the forced transverse root body oscillations as function of the perturbing force amplitude value as well as the soil's elastic deformation and damping coefficients. The ranges of the elastic soil deformation coefficient values, at which the resonant behaviour is observed, that is, at which the forced elastic root body oscillation amplitude value exceeds the tolerance limits, have been determined for the 10, 15 and 20 Hz frequencies of the perturbing force produced by the vibrational lifting tool. That said, the mentioned oscillation amplitude values can vary from 0.58 to 0.45 m, which is sufficient to result in the root breaking. Moreover, it has been proved that, with the increase of the perturbing force frequency, the resonant behaviour ranges shift towards the increased values of the elastic soil deformation coefficient. Therefore, such elastic soil deformation coefficient ranges should be avoided in case of the lifting tool design proposed in the paper. As regards the damping properties of the soil, it has been proved that they do not cause any resonance phenomena.

Key words: frequency, harvester, lifting tool, oscillation amplitude, soil, sugar beet root.

INTRODUCTION

Root-crop harvesters produced in the majority of countries worldwide are generally equipped with vibrational lifting tools, because such tools are capable of lifting roots from the soil without losing or damaging them (Gruber, 2005; Sarec et al., 2009; Gu et al., 2014). Also, the amount of energy required for digging roots from the soil with the use of vibrational lifting tools is significantly lower, than in case of using some other types of tools (Vasilenko et al., 1970; Bulgakov, 2005; Schulze Lammers, 2011).

At the initial stage of the development of vibrational lifting tools for beet harvesters, the forces were applied to the root sitting in the soil in the transverse horizontal plane at right angle to the vector of the lifter's translational motion (Dobrovsky, 1968). However, despite the considerable amount of sufficiently thorough theoretical research into the process of the vibrational sugar beet lifting (Vovk, 1936; Myata & Chumak, 1954), in which the perturbing forces were applied to the roots in the transverse horizontal plane, as well as the numerous engineering developments in that field followed by the industrial scale production of several prototypes and the implementation of comprehensive experimental investigations and official tests, such kind of vibrating digging tools did not gain popularity. The main reason for such an outcome was the fact established in practice that these vibrational lifting tools were unable to support sufficiently high rates of advance (which would, accordingly, provide for higher production rates), while maintaining the required harvesting quality parameters. That, in its turn, was due to the following negative events that took place regularly, when the perturbing forces were applied to the beet roots in the plane that was perpendicular to the line of the lifter's translational motion: the lifter's working throat got clogged with root bodies and soil, the tail parts of roots were broken off, the ability to clear itself was completely lost. The power consumption rate of the process was also unreasonably high (Pogorely & Tatyanko, 2004; Schulze Lammers & Schmittmann, 2013).

Following the above-mentioned unsuccessful attempt, it was found that there was a way to completely avoid those negative events. That could be achieved by changing the lines of action of the perturbing forces, that is, taking them away from the transverse horizontal plane and the perpendicular alignment with respect to the lifter's translational motion and placing them into the longitudinal vertical plane. Such a change resulted in very good indicators, when harvesting sugar beet roots at higher rates of advance. Hence, virtually all worldwide known manufacturers of beet root lifting machinery switched to the production of root-crop harvesters with the vibrational lifting tools that work on the principle of applying perturbing forces to the roots in the longitudinal vertical plane (Gruber, 2007).

In case of the root body performing transverse free and forced oscillations, i.e. when the line of action of the perturbing force coincides with the line of translational motion of the vibrational lifting tool, substantial changes are observed in the process of vibrational root lifting. For example, with such alignment of the perturbing forces, the bonds between the roots and the soil are disrupted more effectively (as a result of the so-called loosening effect) and also the accumulation of roots and soil in the working throat of the vibrational lifting tool becomes significantly reduced. Moreover, the designs of the vibrational lifting tools that work on the above principle are less energy intensive, metal intensive etc. (Boson et al., 2019).

The authors have developed a new vibrational lifting tool (Fig. 1), which during its operation imparts to roots sitting in the soil both longitudinal and transverse oscillations. The structural layout of the proposed digging tool is presented in Fig. 2. The vibrational lifting tool comprises the lifting shares 1 mounted at the ends of the posts 2, which are connected through the suspension brackets 3 with the drive mechanism 4 that sets the above-mentioned shares 1 into oscillatory motion. The mechanism 4 includes the system that allows to set (adjust) the frequency and amplitude of the shares' oscillatory motion within wide ranges (the frequency can be adjusted within 8.0 to 30.0 Hz, the amplitude – within 8 to 24 mm).

A distinctive feature of the proposed vibrational lifting tool is that the bracket 3 used for the suspension of the posts 2 is equipped with an additional hinge, which allows the coupled posts 2 to perform free movement within a small range in the longitudinal transverse plane. Such an arrangement provides for the self-adjustment of the shares 1 during their translational motion in the soil.

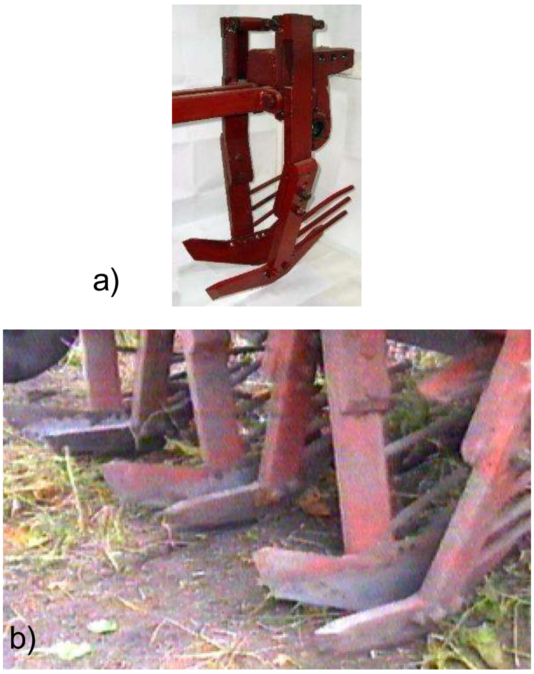

Figure 1. Vibrational lifting tool: a – general appearance; b – tools installed on experimental multiple-row root harvester.

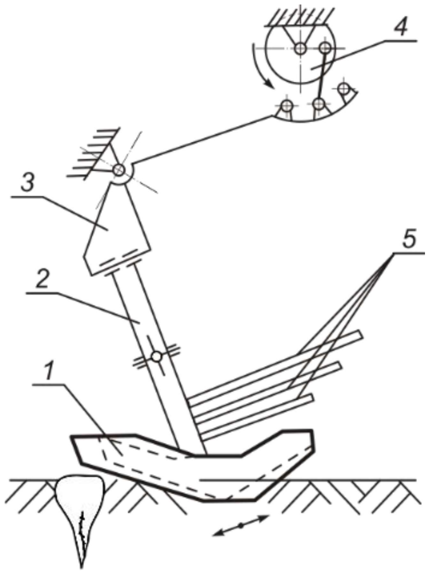

Figure 2. Design and process schematic model of vibrational lifting tool: 1 – lifting shares; 2 – posts; 3 – share spacing adjustment system; 4 – vibration drive with share oscillation amplitude and frequency adjustment system; 5 – guide pins.

However, despite its known considerable advantages, the vibrational method of root lifting features also certain shortcomings. The principal drawbacks are the insufficient reliability (which is true first of all for the vibration drive) that is more pronounced, when operating on heavy and strong soils, the increased metal and energy intensity of the process in general. The inclusion of vibration-type lifting tools, which have a significant weight and oscillate at a frequency of 20 Hz, in the design of the root-crop harvester contributes to the degradation in the reliability of the machine's operation overall.

All the said shortcomings generate the strong need for further development of new principles in the theory of vibrational root lifting, which are to be efficiently utilised for validating the design parameters assumed for further improved lifting tools in root-crop harvesters.

The first fundamental analytical study on the oscillations of the root body fixed in the soil was implemented and presented in the paper Vasilenko et al. (1970). In the study, the sugar beet root was modelled as a conical-shape body with its single lower point fixed, which had elastic properties. With that in view, the paper provided the detailed analysis of the transverse oscillations performed by the root body that were described by a fourth order partial differential equation. The solving of the generated equation provided the authors with the possibility of determining the natural frequencies of the free transverse oscillations performed by the root body fixed in the soil. However, the paper did not provide analytical research into the process of specifically lifting sugar beet roots from the soil, but only stated that the conditions of their lifting had been obtained with the use of additionally generated kinetostatic equations.

In the paper Pogorely et al. (1983), the main principles and assumptions that had effectively been stated already in the paper Vasilenko et al. (1970) were defined and justified. Again, the paper Pogorely et al. (1983) did not present any mathematical model for the vibrational lifting of sugar beet roots from the soil.

The further development of the theory of vibrational root lifting with the perturbing forces applied to the roots specifically in the longitudinal vertical plane can be found in the papers (Vermeulen & Koolen, 2002; Pogorely & Tatyanko, 2004; Bulgakov et al., 2005; Bulgakov & Ivanovs, 2010). However, the case of a root body performing transverse free and forced oscillations, when the perturbing forces are vectored parallel to the translational motion of the vibrational lifting tool, had still not been researched. In the papers Bulgakov et al. (2014, 2015a, 2015b), the development of fundamentals for the theory of free transverse root body oscillations in the case, where the perturbing forces were vectored at right angle to the root's centreline, but parallel to the translational motion of the vibrational lifting tool, was presented.

The aim of this study is to substantiate the parameters of the oscillation process that takes place, when lifting beet roots from the soil, on the basis of developing the theory of the transverse oscillations performed by the root as an elastic body sitting in the soil as an elastic and damping medium for the case, where the perturbing forces have the same direction as the translational motion of the vibrational lifting tool.

MATERIALS AND METHODS

In the completed theoretical research, the fundamental principles of the theory of agricultural machines and the methods of the theoretical mechanics, in particular, the methods of the theory of oscillations, the variational calculus, generation and solution of differential equation systems have been used. In the PC-assisted numerical calculations, the methods of programming and presenting the obtained graphic relations between the main parameters have been applied.

Theory and modelling

In order to investigate the forced transverse oscillations performed by the root sitting in the soil during its vibrational lifting, it is necessary, first of all, to generate the equivalent schematic model of the root as an elastic cone-shaped body sitting in the soil as a medium with elastic and damping properties as well as the external forces applied to the root. Such a schematic model is presented in Fig. 3.

The root as a body has a conical shape (apex angle of the cone is equal to 2γ, its upper part is above the soil surface level) and is modelled as a variable cross-section bar with a fixed lower end (point O). The weight force $\overline{G}$ of the root is applied at the centre of mass represented by the point C. The overall length of the root is designated as h.

The vibrational lifting tool moving at a pre-set depth in the soil (along the vector $\overline{V}$) is conventionally presented as two planes at an angle with each other, which hold the root on its two sides and make contact with it at the points K_1 and K_2. Accordingly, at the above-mentioned points, the vibrational lifting tool imparts to the root the perturbing forces $\bar{Q}_{df.1}$ and $\bar{Q}_{df.2}$, the vectors of which are directed forward and in parallel with each other and which are just the forces that generate the transverse oscillations of the root. These forces are applied at a distance of z_1 from the horizontal line passing through the point *O*.

The soil around the cone-shaped root body is represented by the two elastic and damping models with equal elasticity coefficients of c and damping coefficients of *b*.

The equivalent schematic model is referenced with the Cartesian coordinate system *xOz*, the origin of which is at the point *O* and the vertical axis *Oz* coincides with the axis of symmetry of the cone-shaped root body. The directions of oscillations of both the vibrational lifting tool planes are shown by arrows in the schematic model.

The next step is to analyse the oscillatory process of the transverse oscillations performed by the root body sitting in the soil and generated during the interaction between the root and the vibrating tool.

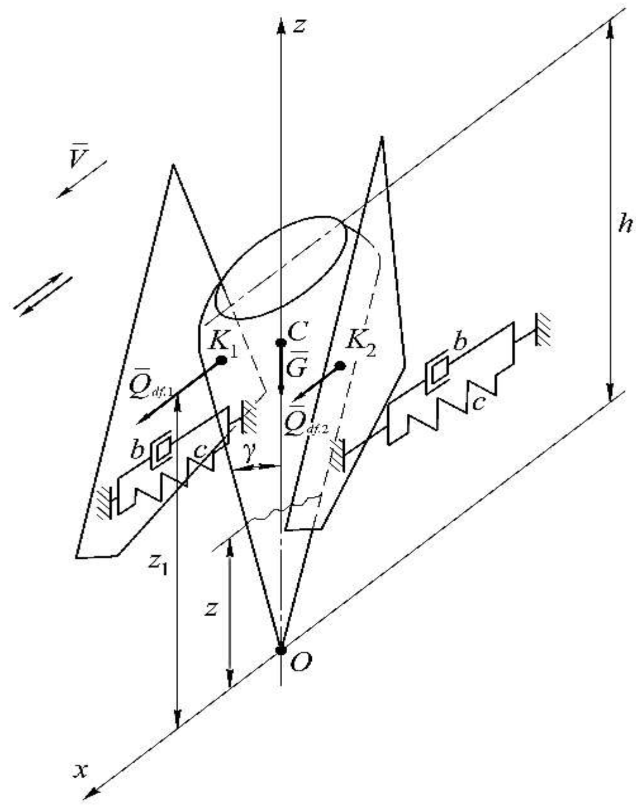

Figure 3. Equivalent schematic model of transverse oscillations performed by root sitting in soil during its vibrational lifting.

The Ostrogradsky-Hamilton principle can be used in the analysis of the forced transverse root oscillations that take place under the action of the horizontal perturbing force that varies in accordance with the following harmonic function (Dreizler & Lüdde, 2010):

$$Q_{df.} = H \cdot \sin(\omega t), \tag{1}$$

where H – amplitude of the perturbing force [N]; ω – frequency of the perturbing force [s^{-1}]; t – time interval [s].

However, as is obvious from the equivalent schematic model (Fig. 3), the above-mentioned perturbing force $\bar{Q}_{df}$ is applied to the root simultaneously on two sides of it by the two digging shares. Therefore, it is represented in the schematic model by the two components $\bar{Q}_{df.1}$ and $\bar{Q}_{df.2}$ and they are exactly what causes the transverse oscillations of the root that disrupt the bonds between the root and the soil and create the conditions needed for lifting the root from the soil.

On the basis of the conditions shown in the prepared equivalent schematic model, the Ostrogradsky-Hamilton functional can be generated, which will provide for analytically describing this kind of root oscillations. Under the above assumptions, the displacements of the root centreline points during the transverse root oscillations are univalently determined by the following function of two variables:

$$y = y(z,t)\,, \tag{2}$$

where z – distance from the point on the axis Oz, through which the root cross-section passes, to the conventional point O of fixing the root in the soil [m]; t – current time [s].

Further, the following designations are introduced.

Thus, $\mu(z)$ – running mass (mass per unit of length) of the root [kg m^{-1}]; E – Young's modulus of the root material [N m^{-2}]; $J(z)$ – moment of inertia of the root cross-section with respect to the cross-section's neutral axis that is perpendicular to the oscillation plane [m^4]; $Q(z,t)$ – intensity of the external transverse load vectored at right angle to the root's centreline (axis Oz) along the axis Ox [N m^{-1}].

In accordance with Babakov (1968), the Ostrogradsky-Hamilton functional for a variable cross-section bar, which performs transverse oscillations under the action of an external transverse load, appears as follows:

$$S = \frac{1}{2}\int_{t_1}^{t_2}\int_{0}^{h}\left[\mu(z)\cdot\left(\frac{\partial y}{\partial t}\right)^2 - E\cdot J(z)\cdot\left(\frac{\partial^2 y}{\partial z^2}\right)^2 + Q(z,t)\cdot y\right]dz\,dt\,. \tag{3}$$

In view of the fact that the root is modelled as a cone-shaped body, the values present in the functional (3) can be expressed in terms of the main parameters of its conical surface.

It is obvious that the running mass of the root can be determined with the use of the following expression:

$$\mu(z) = \rho\cdot\pi\cdot z^2\cdot\tan^2\gamma\,, \tag{4}$$

where ρ – specific gravity of the root material [kg m^{-3}]; 2 γ – taper angle of the cone used as the root body model [deg] (Fig. 3).

The root's moment of inertia $J(z)$ is determined as follows:

$$J(z) = \frac{\pi\cdot z^4\cdot\tan^4\gamma}{4}\,. \tag{5}$$

Since the value Q(z,t) that is a component of the functional (3) is the intensity of a distributed load measured in the N m^{-1} units, the perturbing force $\bar{Q}_{df.}$ that is a concentrated load measured in newtons must also have N m^{-1} as the unit of its measurement. For that purpose, the first-order impulse function $\sigma_1(z)$ is introduced (Babakov, 1968).

Hence, if $Q_{df.}(t)$ is a concentrated perturbing force applied at the point z_1 and measured in newtons [N], the function

$$Q_{df.}(z,t) = Q_{df.}(t)\cdot\sigma_1(z - z_1) \tag{6}$$

has N m^{-1} as the unit of its measurement and represents the intensity of the concentrated load at the point z_1. The function $\sigma_1(z - z_1)$ is equal to zero for all values of z, except $z = z_1$, where it goes to infinity.

Subsequently, taking into account the expression (1), the following can be written down:

$$Q_{df.}(z,t) = H\cdot\sin(\omega t)\cdot\sigma_1(z - z_1)\,. \tag{7}$$

At the start of the oscillatory process, the root is firmly bonded with the soil, the latter being an elastic and damping medium. Therefore, when the perturbing force, the value of which is determined by (1), acts on the root, the force of soil resistance to the transverse root oscillations emerges. Obviously, the soil resistance force (acting on the whole root body) is a load distributed over the area of contact between the root and the soil. Moreover, it is an external force with regard to the root body and it acts as a perturbing force generated by the soil acting on the root.

Further, the parameter *c* is introduced, which is the elastic deformation coefficient of the soil related to the area of contact between the root and the soil, measured in N m^{-3}. It is assumed that the root during its transverse oscillations is supported by the soil on half of its side surface along the whole depth of its sitting in the unbroken soil. The soil contacting with the said half of the side surface generates a distributed load vectored opposite to the perturbing force. Thus, when the root as well as the tool itself perform transverse oscillations, a distributed load, which is applied to the root by the surrounding soil and is opposite in direction to the perturbing force, arises in turn on one side of the root, then on the other side of it, and so on.

Hence, taking into account the above-said and subject to the condition that the root is conically shaped, it is possible to state to some approximation that the intensity $P(z, t)$ [N m^{-1}] of the distributed load generated by the elastic resistance of the soil is equal to:

$$P(z,t) = \pi \cdot c \cdot z \cdot \tan\gamma \cdot y(z,t) \tag{8}$$

In view of the fact that the perturbing forces generated by the vibrational lifting tool and the soil resistance have opposite directions, the resulting intensity of the external transverse load acting on the root has the following value:

$$Q(z,t) = Q_{df.}(z,t) - P(z,t), \ \ [\text{N m}^{-1}], \tag{9}$$

or, taking into account the expressions (7) and (8), the following expression for the soil damping force is arrived at:

$$Q(z,t) = H \cdot \sin(\omega t) \cdot \sigma_1(z - z_1) - \pi \cdot c \cdot z \cdot \tan\gamma \cdot y(z,t), \tag{10}$$

Also, the damping properties of the soil have to be taken into account. They are, first of all: *b* – damping coefficient of the soil measured in (N s^2) m^{-3}.

The following considerations have to be taken into account with regard to the root sitting in the soil at the moment of its lifting. In view of the fact that the root grows and develops its shape in the soil during a considerable length of time, there are good reasons to believe that the area of contact between the whole conical root body and the soil is like a continuous body (it can be said that the root has rather strongly grown into the soil). Therefore, it is also reasonable to assume that great root body deformation rates result in also great deformation rates of the soil around the root. That is due to the certainly great value of the bonding force between the root and the soil enveloping it on all sides, especially in the case, when the root sits in (has grown in) dry and hard soil. Hence, the deformation rate of the soil surrounding the root is virtually equal to the deformation rate of the root body. It is common knowledge that, in case of high rates, the resistance forces follow not linear, but quadratic laws (Schmitz & Smith, 2012). Therefore, it can be assumed to some approximation that the soil damping force is in the quadratic relation with the root body deformation rate.

Therefore, taking into account the conical shape of the root, the soil damping force can be determined with the use of the following expression:

$$R(z,t) = \pi \cdot b \cdot z \cdot \tan\gamma \cdot \left[\frac{\partial y(z,t)}{\partial t}\right]^2, \text{ [N].} \quad (11)$$

Thus, taking into account the expressions (4), (5), (10) and (11), the functional (3) assumes the following form:

$$S = \frac{1}{2}\int_{t_1}^{t_2}\int_{o}^{h}\left[\rho \cdot \pi \cdot z^2 \cdot \tan^2\gamma\left(\frac{\partial y}{\partial t}\right)^2 - E \cdot \frac{\pi \cdot z^4 \cdot \tan^4\gamma}{4}\left(\frac{\partial^2 y}{\partial z^2}\right)^2 + \right.$$
$$\left. + H \cdot \sin(\omega t) \cdot \sigma_1(z - z_1) \cdot y(z,t) - \pi \cdot c \cdot z \cdot \tan\gamma \cdot y^2(z,t) - \pi \cdot b \cdot z \cdot \tan\gamma \cdot \left(\frac{\partial y}{\partial t}\right)^2\right] dz\,dt. \quad (12)$$

In order to analyse the forced transverse oscillations of the root body fixed in the soil, the Rietz method can be applied (Babakov, 1968).

In view of the fact that the perturbing force acts on the root at a frequency of ω, its solely forced oscillations occur in accordance with the following function (Babakov, 1968):

$$y(z,t) = \varphi(z)\sin(\omega t), \quad (13)$$

where $\varphi(z)$ – waveform of the forced oscillations.

The necessary partial derivatives have to be derived from the expression (13). They appear as follows:

$$\frac{\partial y}{\partial t} = \omega \cdot \varphi(z) \cdot \cos(\omega t),$$
$$\frac{\partial^2 y}{\partial z^2} = \varphi''(z) \cdot \sin(\omega t). \quad (14)$$

By substituting the expressions (13) and (14) into the functional (12), the latter's following representation is arrived at:

$$S = \frac{1}{2}\int_{t_1}^{t_2}\int_{o}^{h}\left\{\rho \cdot \pi \cdot z^2 \cdot \tan^2\gamma \cdot \omega^2 \cdot \varphi^2(z) \cdot \cos^2(\omega t) - \frac{E \cdot \pi \cdot z^4 \cdot \tan^4\gamma}{4}\left[\varphi''(z)\right]^2 \cdot \sin^2(\omega t) + \right.$$
$$+ H \cdot \sigma_1(z - z_1) \cdot \varphi(z) \cdot \sin^2(\omega t) - \pi \cdot c \cdot z \cdot \tan\gamma \cdot \varphi^2(x) \cdot \sin^2(\omega t) - \quad (15)$$
$$\left. - \pi \cdot b \cdot z \cdot \tan\gamma \cdot \omega^2 \cdot \varphi^2(z) \cdot \cos^2(\omega t)\right\} dz\,dt.$$

After integrating the expression (15) over z within the limits of one period, that is, within $T = \frac{2\pi}{\omega}$, the result is:

$$S = \frac{\pi}{2\omega}\int_{o}^{h}\left\{\rho \cdot \pi \cdot z^2 \cdot \tan^2\gamma \cdot \varphi^2(z) \cdot \omega^2 - \frac{E \cdot \pi \cdot z^4 \cdot \tan^4\gamma}{4} \cdot \left[\varphi''(z)\right]^2 + \right.$$
$$\left. + H \cdot \sigma_1(z - z_1) \cdot \varphi(z) - \pi \cdot c \cdot z \cdot \tan\gamma \cdot \varphi^2(z) - \pi \cdot b \cdot z \cdot \tan\gamma \cdot \omega^2 \cdot \varphi^2(z)\right\} dz. \quad (16)$$

In accordance with the Rietz method, the values of the functional (16) have to be analysed on the class of linear combinations defined as follows:

$$\varphi(z) = \alpha \cdot \psi(z), \tag{17}$$

where α – parameter, the variation of which produces a class of admissible functions; $\Psi(z)$ – basis function.

After the expression (17) is substituted into the functional (16), the following is arrived at:

$$S = \frac{\pi}{2\omega}\int_o^h \Big\{ \rho \cdot \pi \cdot z^2 \cdot \tan^2\gamma \cdot \alpha^2 \cdot \psi^2(z) \cdot \omega^2 - \frac{E \cdot \pi \cdot z^4 \cdot \tan^4\gamma}{4}\alpha^2\left[\psi''(z)\right]^2 + \\ + H \cdot \sigma_1(z - z_1) \cdot \alpha \cdot \psi(z) - \pi \cdot c \cdot z \cdot \tan\gamma \cdot \alpha^2 \cdot \psi^2(z) - \pi \cdot b \cdot z \cdot \tan\gamma \cdot \alpha^2 \cdot \psi^2(z) \cdot \omega^2 \Big\} dz. \tag{18}$$

Next, the following designations have to be introduced:

$$\int_o^h \rho \cdot \pi \cdot z^2 \cdot \tan^2\gamma \cdot \psi^2(z) dz = M, \tag{19}$$

$$\int_0^h \frac{E \cdot \pi \cdot z^4 \cdot \tan^4\gamma}{4}\left[\psi''(z)\right]^2 dz = N, \tag{20}$$

$$\int_o^h \pi \cdot c \cdot z \cdot \tan\gamma \cdot \psi^2(z) dz = R, \tag{21}$$

$$\int_o^h \pi \cdot b \cdot z \cdot \tan\gamma \cdot \psi^2(z) dz = F, \tag{22}$$

$$\int_o^h H \cdot \sigma_1(z - z_1) \cdot \psi(z) dz = L. \tag{23}$$

After substituting the expressions (19) – (23) into (18), the result is:

$$S = \frac{\pi}{2\omega}\left[\omega^2 \cdot (M - F) \cdot \alpha^2 - (N + R) \cdot \alpha^2 + L \cdot \alpha\right]. \tag{24}$$

Thus, on the class of functions (17), the functional (18) becomes a function of the free variable α. The necessary extremum condition of the function (24) is that its first-order derivative with respect to α is equal to zero. Accordingly, after differentiating the expression (24) with respect to α and equating the obtained derivative to zero, the following equation is arrived at:

$$2\omega^2 \cdot (M - F) \cdot \alpha - 2(N + R) \cdot \alpha - L = 0, \tag{25}$$

from which the parameter α can be determined, that is:

$$\alpha = \frac{L}{2\left[N + R - \omega^2 \cdot (M - F)\right]}. \tag{26}$$

Further, the waveform of the forced transverse oscillations performed by a homogeneous bar with a constant elastic stiffness of *EJ*, one end of which is rigidly fixed, under the action of a transverse harmonic unit force with a frequency of ω applied at the point $z = z_1$ is assumed to be the basis function $\Psi(z)$. In accordance with (Babakov, 1968), this waveform appears as follows:

$$\psi(z) = C \cdot U(kz) + D \cdot V(kz), \quad 0 \le z < z_1, \tag{27}$$

$$\psi(z)=C\cdot U(kz)+D\cdot V(kz)+\frac{1}{k^3\cdot EJ}\cdot V\left[k\left(z-z_1\right)\right],\quad z_1\le z\le h\,, \tag{28}$$

where $U(kz)$, $V(kz)$ – Krylov functions (Babakov, 1968), $k=\sqrt[4]{\frac{\mu\cdot\omega^2}{EJ}}$ – coefficient representing the mass, kinematic and strength properties of the bar's material, μ – running mass of the bar; C, D – arbitrary constants. At the same time, the boundary conditions for the oscillations performed by the above-mentioned bar are as follows:

$$\begin{aligned} &y(0)=y'(0)=0,\\ &y''(h)=y'''(h)=0. \end{aligned} \tag{29}$$

Taking into account the said boundary conditions (29) for the free end of the bar ($z=h$), the following system of equations with respect to the unknown quantities $\Psi(z),\ C, D$ is obtained:

$$\left.\begin{aligned} &-\psi(z)+C\cdot U(kz)+D\cdot V(kz)=\begin{cases}0, & 0\le z<z_1,\\ -\frac{1}{k^3\cdot EJ}\cdot V\left[k\left(z-z_1\right)\right], & z_1\le z\le h,\end{cases}\\ &C\cdot S(kh)+D\cdot T(kh)=-\frac{1}{k^3\cdot EJ}T\left[k\left(h-z_1\right)\right],\\ &C\cdot V(kh)+D\cdot S(kh)=-\frac{1}{k^3\cdot EJ}S\left[k\left(h-z_1\right)\right]. \end{aligned}\right\} \tag{30}$$

For the problem under consideration, it is necessary to determine only such a basis function $\Psi(z)$ that meets the above-mentioned boundary conditions. Using the Cramer's rule, the target value of $\Psi(z)$ is obtained from the system of Eqs (30). It is equal to:

$$\begin{aligned} \psi(z)=&\frac{U(kz)}{\Delta\cdot k^3\cdot EJ}\left\{T\left[k\left(h-z_1\right)\right]S(kh)-T(kh)S\left[k\left(h-z_1\right)\right]\right\}-\\ &-\frac{V(kz)}{\Delta\cdot k^3\cdot EJ}\left\{T\left[k\left(h-z_1\right)\right]\cdot V(kh)-S\left[k\left(h-z_1\right)\right]S(kh)\right\},\quad 0\le z<z_1\,, \end{aligned} \tag{31}$$

or

$$\begin{aligned} \psi(z)=&-\frac{V\left[k\left(z-z_1\right)\right]}{\Delta\cdot k^3\cdot EJ}\left\{S^2(kh)-T(kh)\cdot V\cdot(kh)\right\}+\\ &+\frac{U(kz)}{\Delta\cdot k^3\cdot EJ}\left\{T\left[k\left(h-z_1\right)\right]\cdot S(kh)-T(kh)\cdot S\left[k\left(h-z_1\right)\right]\right\}-\\ &-\frac{V(kz)}{\Delta\cdot k^3\cdot EJ}\left\{T\left[k\left(h-z_1\right)\right]\cdot V(kh)-S\left[k\left(h-z_1\right)\right]\cdot S(kh)\right\},\quad z_1\le z\le h\,, \end{aligned} \tag{32}$$

where Δ – principal determinant of the system of Eqs (30) equal to:

$$\Delta=T(kh)\cdot V(kh)-S^2(kh)\,. \tag{33}$$

Further, the following designations have to be introduced:

$$\frac{T\left[k\left(h-z_1\right)\right]\cdot S\left(kh\right)-T\left(kh\right)\cdot S\left[k\left(h-z_1\right)\right]}{\Delta\cdot k^3\cdot EJ}=B\,, \tag{34}$$

also:

$$\frac{T\left[k\left(h-z_1\right)\right]\cdot V\left(kh\right)-S\left[k\left(h-z_1\right)\right]\cdot S\left(kh\right)}{\Delta\cdot k^3\cdot EJ}=G\,, \tag{35}$$

finally:

$$\frac{S^2\left(kh\right)-T\left(kh\right)\cdot V\left(kh\right)}{\Delta\cdot k^3\cdot EJ}=K\,. \tag{36}$$

By substituting the expressions (34), (35), (36) into the expressions (31), (32), the following is obtained:

$$\psi\left(z\right)=B\cdot U\left(kz\right)-G\cdot V\left(kz\right),\qquad 0\le z<z_1\,, \tag{37}$$

$$\psi\left(z\right)=-K\cdot V\left[k\left(z-z_1\right)\right]+B\cdot U\left(kz\right)-G\cdot V\left(kz\right),\qquad z_1\le z\le h\,. \tag{38}$$

The next step is to determine the coefficients *M, N, R, F, L* that are present in the expressions (26).

For that purpose, the expressions (37), (38) are substituted into the expressions (19) and the value of the coefficient M is found. It is equal to:

$$\begin{aligned}M&=\rho\cdot\pi\cdot\tan^2\gamma\cdot\int_0^{z_1}z^2\cdot\left[B\cdot U\left(kz\right)-G\cdot V\left(kz\right)\right]^2dz+\\&+\rho\cdot\pi\cdot\tan^2\gamma\cdot\int_{z_1}^{h}z^2\cdot\left\{-K\cdot V\left[k\left(z-z_1\right)\right]+B\cdot U\left(kz\right)-G\cdot V\left(kz\right)\right\}^2dz.\end{aligned} \tag{39}$$

In order to determine the coefficient *N*, the second derivatives of the expressions (37), (38) are obtained as follows:

$$\psi''\left(z\right)=B\cdot k^2\cdot S\left(kz\right)-G\cdot k^2\cdot T\left(kz\right),\qquad 0\le z<z_1, \tag{40}$$

$$\psi''\left(z\right)=-K\cdot k^2\cdot T\left[k\left(z-z_1\right)\right]+B\cdot k^2\cdot S\left(kz\right)-G\cdot k^2\cdot T\left(kz\right),\qquad z_1\le z\le h\,. \tag{41}$$

After the expressions (37), (38) are substituted into the expression (20), the value of the coefficient N is found as follows:

$$\begin{aligned}N&=\frac{E\cdot\pi\cdot\tan^4\gamma}{4}\cdot\int_0^{z_1}k^4\cdot z^4\cdot\left[B\cdot S\left(kz\right)-G\cdot T\left(kz\right)\right]^2dz+\\&+\frac{E\cdot\pi\cdot\tan^4\gamma}{4}\cdot\int_{z_1}^{h}k^4\cdot z^4\cdot\left\{-K\cdot T\left[k\left(z-z_1\right)\right]+B\cdot S\left(kz\right)-G\cdot T\left(kz\right)\right\}^2dz\,.\end{aligned} \tag{42}$$

By substituting the expressions (40), (41) into the expression (21), the value of the coefficient *R* is obtained as follows:

$$\begin{aligned}R&=c\cdot\pi\cdot\tan\gamma\cdot\int_0^{z_1}z\cdot\left[B\cdot U\left(kz\right)-G\cdot V\left(kz\right)\right]^2dz+\\&+c\cdot\pi\cdot\tan\gamma\cdot\int_{z_1}^{h}z\cdot\left\{-K\cdot V\left[k\left(z-z_1\right)\right]+B\cdot U\left(kz\right)-G\cdot V\left(kz\right)\right\}^2dz.\end{aligned} \tag{43}$$

By substituting the same expressions into the expression (22), the value of the coefficient F is found as follows:

$$\begin{aligned} F &= b\cdot\pi\cdot\tan\gamma\cdot\int_0^{z_1} z\cdot\left[B\cdot U(kz)-G\cdot V(kz)\right]^2 dz+ \\ &+b\cdot\pi\cdot\tan\gamma\cdot\int_{z_1}^{h} z\cdot\left\{-K\cdot V\left[k(z-z_1)\right]+B\cdot U(kz)-G\cdot V(kz)\right\}^2 dz. \end{aligned} \tag{44}$$

The coefficient L is obtained by substituting the expressions (37), (38) into the expression (23). Its value is equal to:

$$\begin{aligned} L &= \int_0^{z_1} H\cdot\sigma_1(z-z_1)\cdot\left[B\cdot U(kz)-G\cdot V(kz)\right]dz+ \\ &+\int_{z_1}^{h} H\cdot\sigma_1(z-z_1)\cdot\left\{-K\cdot V\left[k(z-z_1)\right]+B\cdot U(kz)-G\cdot V(kz)\right\}dz. \end{aligned} \tag{45}$$

The numerical values of the coefficients *M, N, R* and *F* can be calculated with the use of the PC either by directly taking integrals of the Krylov functions or after the transition to elementary functions in accordance with (Babakov, 1968).

In view of the fact that the expression (45) intended for finding the coefficient L contains the impulse function $\sigma_1(z-z_1)$ that does not fall into the category of classical functions, the integrals present in the said expression have to be calculated analytically with the use of the generalised function integration technique.

As a result of integrating the expression (45), the following is obtained:

$$L = H\left[B\cdot U(kz_1)-G\cdot V(kz_1)\right]. \tag{46}$$

Then, after substituting the expressions (39), (42), (43), (44) and (46) into the expression (26), the required value of the parameter α, at which the functional (16) has a stationary value, is obtained. Respectively, taking into account the expressions (17), (37) and (38), the expressions that represent the waveform of the forced transverse oscillations of the root body fixed in the soil are obtained.

These expressions appear as follows:

$$\varphi(z) = \alpha\cdot\left[B\cdot U(kz)-G\cdot V(kz)\right],\quad 0\le z<z_1, \tag{47}$$

$$\varphi(z) = \alpha\cdot\left\{-K\cdot V\left[k(z-z_1)\right]+B\cdot U(kz)-G\cdot V(kz)\right\},\quad z_1\le z\le h \tag{48}$$

where α is determined by the expression (26).

The substitution of the expressions (47) and (48) into the expression (13) results in obtaining the final representation of the function governing the forced transverse oscillations of the root body fixed in the soil:

$$y(z,t) = \alpha\cdot\left[B\cdot U(kz)-G\cdot V(kz)\right]\cdot\sin(\omega t),\quad 0\le z<z_1, \tag{49}$$

$$y(z,t) = \alpha\cdot\left\{-K\cdot V\left[k(z-z_1)\right]+B\cdot U(kz)-G\cdot V(kz)\right\}\cdot\sin(\omega t),\quad z_1\le z\le h. \tag{50}$$

RESULTS

Following the results of the theoretical research into the forced transverse root body oscillations described above, the authors developed the algorithm for calculating the said oscillations.

The following initial data has been used in the calculations. In accordance with (Vasilenko et al., 1970), the ranges of the soil's elastic stiffness and damping coefficients were assumed to be as follows: b = 0–10 N s^2 m^{-3}, c = 0–20 105 N m^{-3}.

In accordance with (Pogorely et al., 1983), the average statistic values of the root's physical and mechanical properties were assumed to be as follows: h = 0.25 m; $\gamma = 14°$; E = 18.4 10^6 N m^{-2}; ρ = 750 kg m^{-3}.

Calculations were carried out with the use of the compiled programme. Their results were used to plot the diagrams that represented the relations between the amplitude of the forced transverse oscillations performed by the root body as an elastic body, which sits (in practice, is fixed) in the soil as an elastic and damping medium, and the amplitude H of the perturbing force as well as the mechanical properties of the soil surrounding it. The obtained diagrams are presented in Figs 4–7.

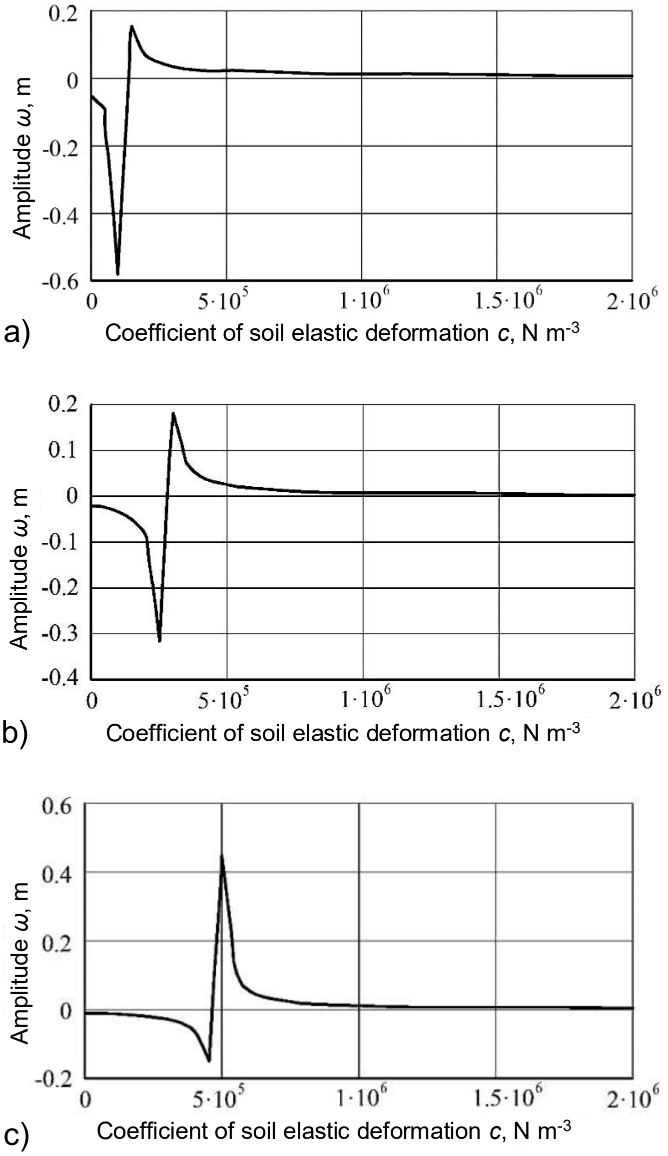

Figure 4. Relation between amplitude of forced transverse root body oscillations on one hand and coefficient of elastic soil deformation c and perturbing force frequency v on the other hand for root's cross-section at point of its gripping ($z = z_1 = 0.15$ m): a) ν = 10 Hz; b) ν = 15 Hz; c) ν = 20 Hz (perturbing force amplitude H = 500 N, b = 6.5 N s^2 m^{-3}, c = 0–20 N m^{-3}).

Fig. 4 features the diagrams of the relations between the amplitude of the forced transverse elastic root body oscillations and the coefficient c of the elastic soil deformation at various perturbing force frequencies: 10 Hz, 15 Hz, 20 Hz. The calculations have been carried out for a perturbing force amplitude of H = 500 N and a soil damping coefficient of b = 6.5 N s^2 m^{-3}.

As is seen in the presented diagrams, the resonance state takes place at the following values of the parameters: $\nu = 10$ Hz, c = 1.0–1.3·10^5 N m^{-3}; $\nu = 15$ Hz, $c = 2.5$–$2.7 \cdot 10^5$ N m^{-3}; $\nu = 20$ Hz, $c = 4.5$–$5.0 \cdot 10^5$ N m^{-3}.

Moreover, at $\nu = 10$ Hz the amplitude of the root body oscillations in the resonance state varies within the range of – 0.58 to 0.17 m, at $\nu = 15$ Hz within the range of – 0.31 to 0.18 m, at $\nu = 20$ Hz within the range of – 0.17 to 0.45 m.

That said, it ought to be noted that the resonance state range shifts to the right with the increase of the perturbing force frequency, that is, the resonance takes place at higher values of the elastic soil deformation coefficient c.

Hence, within the above-mentioned resonance state ranges of the root body oscillation parameters, the chipping off of the root, especially in its tail part, can take place.

In Fig. 5, the diagrams are shown for the relations between the amplitude of the forced transverse root body oscillations on the one hand and the elastic soil deformation coefficient c and the distance z from the root's cross-section to the conventional point of its fixing in the soil O on the other hand at a perturbing force amplitude of $H = 500$ N and a damping coefficient of $b = 6.5$ N s^2 m^{-3}.

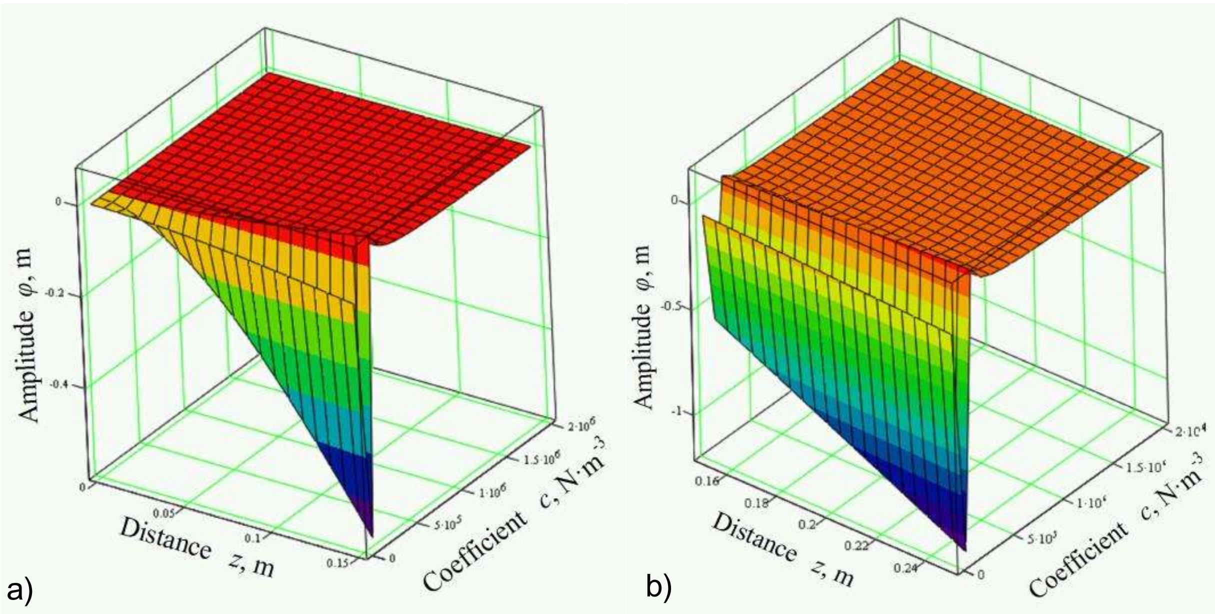

Figure 5. Relation between amplitude of forced transverse root body oscillations on one hand and coefficient of elastic soil deformation c and distance from root's cross-section to conventional point of its fixing z on the other hand: a) $z = 0$–0.15 m; b) $z = 0.15$–0.25 m; (perturbing force amplitude $H = 500$ N, $b = 6.5$ N s^2 m^{-3}, perturbing force frequency $v = 10$ Hz).

As is obvious from the above diagrams, the amplitude of the forced transverse oscillations sharply rises at $c = 1.0 \cdot 10^5 \ldots 1.3 \cdot 10^5$, N m^3 and $z = 0.15\ m$, $z = 0.5\ m$, that is, at the point, where the tool grips the root and at the end of the root.

Also, in these cases the frequency of forced oscillations is virtually equal to the frequency of free oscillations of the root body. Within this range of parameters of the elastic soil deformation, the amplitude can reach up to 0.58 m. At all other values of the elastic soil deformation coefficients, the amplitude is close to zero and stays within the range of several millimetres.

Fig. 6 features the graphic relations between the amplitude of the forced transverse root body oscillations and the soil damping coefficient b at perturbing force frequencies ν equal to 10 Hz, 15 Hz, 20 Hz, a perturbing force amplitude of $H = 500$ N and an elastic soil deformation coefficient of $c = 2.0 \cdot 10^5$, N m^{-3}.

As is seen in the diagrams, at $\nu = 10$ Hz and when the elastic deformation coefficient b varies within the range of 0 to 10 N s^2 m^{-3}, the amplitude of the forced transverse oscillations decreases from 0.088 to 0.056 m; at $\nu = 15$ Hz, it increases from 0.050 to 0.175 m; at $\nu = 20$ Hz, it increases from 0.015 to 0.025 m.

However, at $b = 6.5$ N s^2 m^{-3}, which is the most common value of the soil damping coefficient, the amplitudes in the above cases are equal to:

at $\nu = 10$ Hz – 0.063 m;

at $\nu = 15$ Hz – 0.085 m;

at $\nu = 20$ Hz – 0.020 m,

which is within the range of the permissible root body deformations.

As is seen in the diagrams presented in Fig. 6, the resonance phenomena in these cases do not take place.

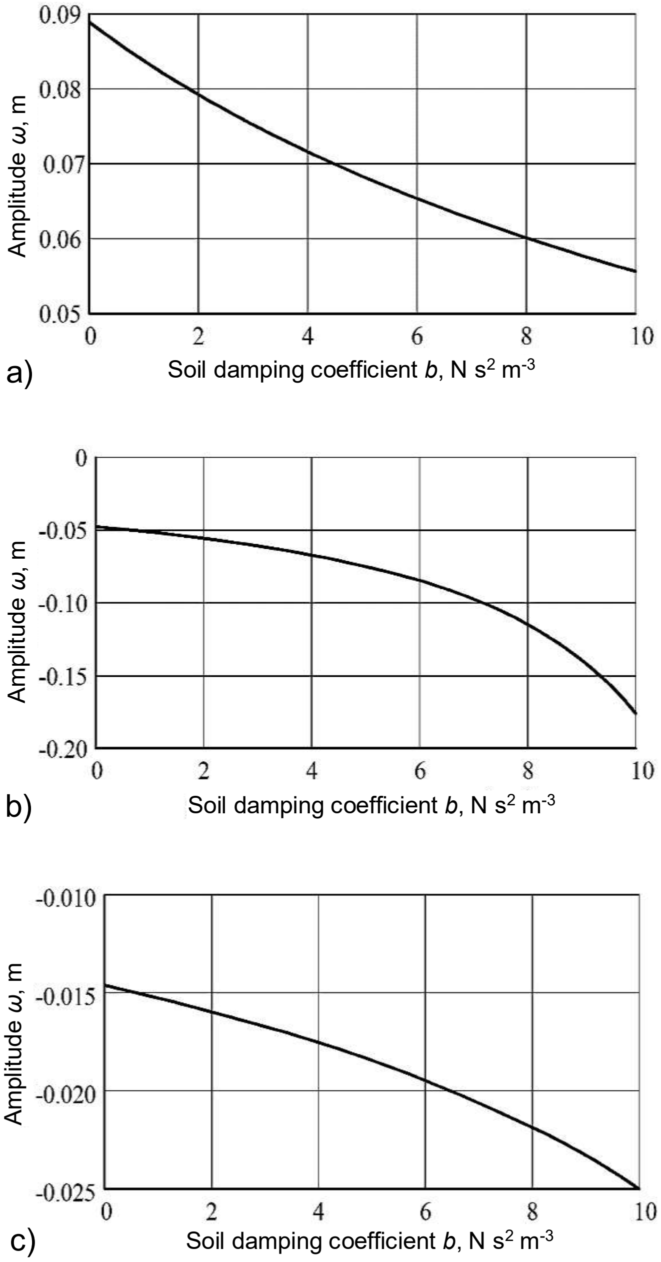

Figure 6. Relation between amplitude of forced transverse root body oscillations on one hand and soil damping coefficient b and perturbing force frequency ν on the other hand for root's cross-section at point of its gripping by tool ($z = z_1 = 0.15$ m): a) $\nu = 10$ Hz; b) $\nu = 15$ Hz; c) $\nu = 20$ Hz (perturbing force amplitude $H = 500$ N, $b = 0$–10 N s^2 m^{-3}, $c = 2\ 10^5$ N m^3).

According to the diagrams shown in Fig. 7, the maximum amplitude of the transverse oscillations is reached at the point, where the tool grips the root (z = 0.15 m), and at the end of the root (z = 0.24 m), but again, resonance phenomena are not observed in these cases.

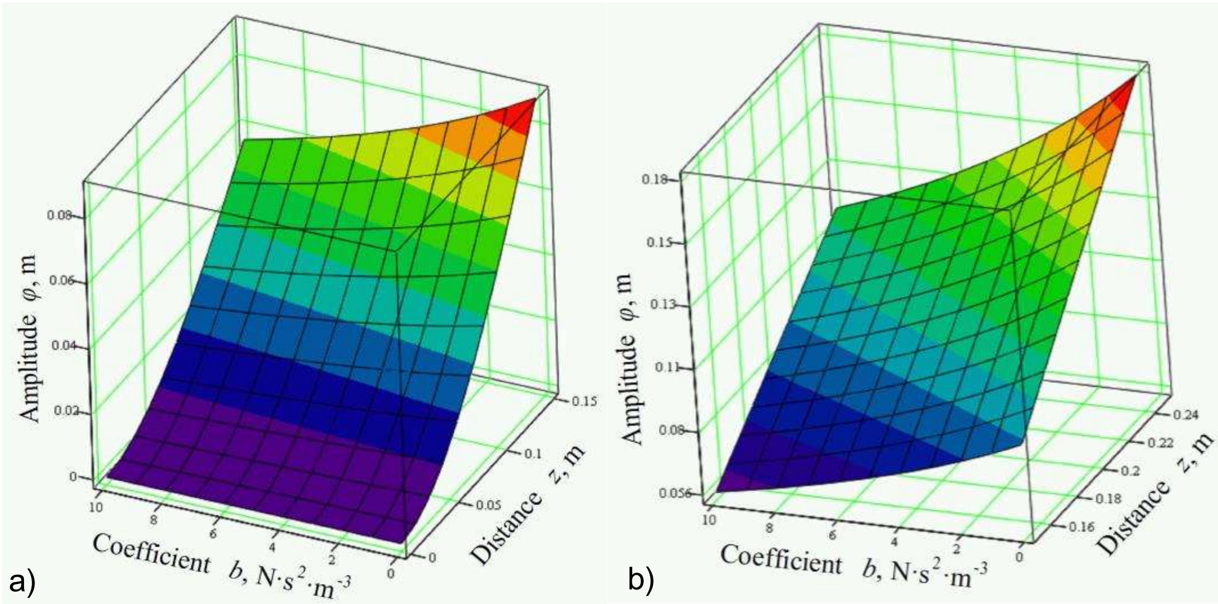

Figure 7. Relation between amplitude of forced transverse root body oscillations on one hand and soil damping coefficient b and distance from root's cross-section to conventional point of its fixing z on the other hand: a) z = 0–0.15 m; b) z = 0.15–0.25 m; (perturbing force amplitude H = 500 N, perturbing force frequency ν = 10 Hz).

Thereby, by using the results of the PC-assisted calculations, the ranges of values have been obtained for the elastic soil deformation coefficient, within which the resonance state takes place. That is, the value of the amplitude of the forced transverse oscillations performed by the elastic root body exceeds the acceptable limit values.

CONCLUSIONS

1. The fundamental principles have been developed for the theory of the transverse oscillations performed by the root as an elastic body sitting in the soil as an elastic and damping medium during its vibrational lifting in the case, where the perturbing forces are vectored the same as the translational motion of the digging tool.
2. By using the generated equivalent schematic model and applying the Ostrogradsky-Hamilton variational principle, the analytical expressions have been obtained for calculating the amplitude of the forced transverse root body oscillations at any cross-section of the root.
3. The specially developed computer programme has been used to carry out the PC-assisted numerical calculations, which have provided for plotting the diagrams showing how the amplitude of the forced transverse oscillations of the root as an elastic body sitting in the soil as an elastic and damping medium varies in relation to the amplitude of the perturbing force and the coefficients of elastic soil deformation and soil damping.

4. The results of the calculations have been used to find the ranges of the elastic soil deformation coefficient values, within which the resonance state takes place. That is, the value of the amplitude of the forced transverse elastic root body oscillations exceeds the permissible limits, for the cases, when the frequency of the perturbing force generated by the vibrational lifting tool is equal to $v = 10$, 15 and 20 Hz. Under such conditions, the above-mentioned amplitudes of oscillations can vary within the range of 0.45 to 0.58 m, especially at the points of gripping the root and at the end of the root. That can result in chipping off the root's end, therefore, such ranges of the soil's elastic stiffness should be avoided.

5. When the soil damping coefficient b varies within the sufficiently wide range of 0–10 N s^2 m^{-3}, resonance phenomena are absent, the amplitudes of transverse oscillations stay within permissible limits. Therefore, the damping properties of the soil are acceptable within the whole range under consideration.

6. When engineering the lifting tools with the discussed direction of oscillations, it is necessary to take into account the elastic properties of the soil, with which such tools can operate efficiently.

7. The results of the completed analytical research have been used in the development of a new design of vibrational lifting tools.

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